1. Prove or disprove the following:
   (a) For all \( m, n \in \mathbb{Z} \), if \( mn \) are odd, then \( m \) and \( n \) are odd.
   Consider the contrapositive: If \( m \) or \( n \) are even, then \( m \times n \) is even.
   By definition the product is even in each of the three cases:
   i. \( m \) is even and \( n \) is odd:
   \[ m = 2a, \quad n = 2b + 1 \]
   \[ m \times n = 2a(2b + 1) = 2(ab + a) \Rightarrow m \times n \text{ is even.} \]
   ii. \( m \) is even and \( n \) is even:
   \[ m = 2a, \quad n = 2b \]
   \[ m \times n = 2a \times 2b = 4ab = 2(2ab) \Rightarrow m \times n \text{ is even.} \]
   iii. \( m \) is odd and \( n \) is even:
   \[ m = 2a + 1, \quad n = 2b \]
   \[ m \times n = 2a(2b + 1) - 1 = 4ab + 2b - 1 \]
   \[ m \times n \text{ is even.} \]
   Therefore, \( m \) and \( n \) are odd.

(b) There are no integer solutions to the equation \( x^2 + 19 = y^2 + 2021 \).
   We know that \((x + y)(x - y) = 2002\) so \((x + y)(x - y)\) is even. Therefore one of \((x + y)\) and \((x - y)\) must be odd. The sum of an even and odd number is odd, which implies \((x + y) + (x - y)\) must be odd, but \((x + y) + (x - y) = 2x\) which is even. This is a contradiction. Therefore no integer solutions.

(c) Let \( n \in \mathbb{N}, n > 1 \). If \( n \) is not prime then \( 2^n - 1 \) is not prime. Hint: you may use the identity \((a^x - 1) = (a-1)(a^{x-1} + a^{x-2} + \cdots + a^1 + 1)\).
   If \( n \) is not prime, then \( n \) is composite and can be written as \( n = \beta \times \gamma \) with \( 1 < \beta, \gamma < n \):
   \[ 2^n - 1 = 2^{\gamma \beta} - 1 = (2^{\gamma} - 1)(2^{\gamma(\beta-1)} + 2^{\gamma(\beta-2)} + \cdots + 2^{\gamma} + 1) \]
   In the above, \( \gamma > 1 \Rightarrow (2^{\gamma} - 1) > 1 \) and \( \beta > 1 \Rightarrow (2^{\gamma(\beta-1)} + 2^{\gamma(\beta-2)} + \cdots + 2^{\gamma} + 1) < 2^{n-1} \). Therefore \( 2^n - 1 \) is composite.

2. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.
   (a) For all \( n \in \mathbb{N}, \sum_{i=1}^{n} (2i - 1) = n^2 \)
   Base case when \( n = 1 : (2 \times 1 - 1) = 1^2 \)
   Inductive hypothesis: Assume \( \sum_{i=1}^{k} (2i - 1) = k^2 \)
   Inductive step:
   \[
   \sum_{i=1}^{k+1} (2i - 1) = \sum_{i=1}^{k} (2i - 1) + 2(k + 1) - 1
   \]
   \[ = k^2 + 2k + 1 \]
   \[ = (k + 1)^2 \]

(b) For all \( n \in \mathbb{N}, n \geq 4, 3^n \geq n^3 \)
   Base case when \( n = 4 : 3^4 = 81 \geq 4^3 = 64 \)
Inductive hypothesis: Assume $n \geq 4$, $3^n \geq n^3$ for some $k \geq 4$

Inductive step:

$$3^{k+1} = 3^1 \times 3^k$$

$$\geq k^3 + k^3 + k^3 \text{ (by IH)}$$

$$\geq k^3 + 3k^2 + 3k^2 \text{ (since } k \geq 4)$$

$$= k^3 + 3k^2 + k^2 + k^2 \text{ (expanding the last term)}$$

$$\geq k^3 + 3k^2 + 3k + 3k + 3k \text{ (since } k \geq 4)$$

$$\geq k^3 + 3k^2 + 3k + 1 \text{ (since } k \geq 4)$$

$$= (k + 1)^3$$

(c) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

Base case when $n = 1$: $\frac{1}{1(1+1)} = \frac{1}{2}$.

Inductive hypothesis: Assume $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ true for some $k$.

Inductive step:

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} \text{ by IH}$$

$$= \frac{k(k+2) + 1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

3. Use the Euclidean Algorithm to find the greatest common divisor of 143 and 33. Use the Extended Euclidean Algorithm to find the Bezout coefficients $x, y$ and all integer solutions to the equation $143x + 33y = \gcd(143, 33)$.

$$143 = 4 \times 33 + 11 \Rightarrow 11 = 143 - 4 \times 33$$

$$33 = 3 \times 11 + 0$$

Therefore the $\gcd(143, 33) = 11$ and $x = 1, y = -4$. Using the identity $\text{lcm}(a, b) = \frac{ab}{\gcd(a, b)}$ we find that:

$$143 \times \frac{33}{11} = 33 \times \frac{143}{11}$$

$$143 \times 3 = 33 \times 13$$

$$143 \times 3 - 33 \times 13 = 0$$

$$143 \times 3k + 33 \times (-13k) = 0$$

Adding the above with Bezout’s identity:

$$143 \times (3k + 1) + 33 \times (-13k - 4) = 11$$
4. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.

(a) For all \( n \in \mathbb{N}, n \geq 2, \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n\cdot2}. \)

Base case when \( n = 2: \prod_{i=2}^{2} \left(1 - \frac{1}{i^2}\right) = \frac{3}{4} = \frac{2+1}{4}. \)

Inductive hypothesis: Assume \( \prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k} \) true for some \( k. \)

Inductive step:

\[
\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) \times \left(1 - \frac{1}{(k+1)^2}\right)
\]

\[
= \frac{(k+1)}{2k} \times \frac{(k+1)(k+2)}{2(k+1)(k+1)}
\]

\[
= \frac{k+2}{2(k+1)} \quad \square
\]

(b) For all \( n \in \mathbb{N}, 7|(2^{n+2} + 3^{2n+1}) \)

Base case when \( n = 1: 7|(2^3 + 3^3) \Rightarrow 7|35 \)

Inductive hypothesis: Assume \( 7|(2^{k+2} + 3^{2k+1}) \) for some \( k. \)

Inductive step:

\[
2^{k+3} + 3^{2k+2} = (9 - 7) \times 2^{k+3} + 9 \times 3^{2k+1}
\]

\[
= 9 \left( \frac{2^{k+2}}{7} + \frac{3^{2k+1}}{7}\right) - 7 \times \frac{2^{k+2}}{7} \times \frac{3^{2k+1}}{7}
\]

Therefore \( 9(2^{k+2} + 3^{2k+1}) - 7 \times 2^{k+2} \) is divisible by 7 \( \square \)

5. Verify whether the following functions define bijections.

(a) Let \( A = \{x \in \mathbb{R} : x \neq -2\} \) and \( B = \{x \in \mathbb{R} : x \neq 1\} \) is the function \( f : A \rightarrow B \) defined by \( f(x) = \frac{x-2}{x+2} \) (i) injective (ii) surjective and (iii) bijective? Prove or disprove.

i. \( f(x_1) = f(x_2) \Rightarrow \frac{x_1-2}{x_1+2} = \frac{x_2-2}{x_2+2} \Rightarrow (x_1-2)(x_2+2) = (x_2-2)(x_1+2) \Rightarrow x_1x_2 - 2x_2 + 2x_1 - 4 = x_1x_2 + 2x_2 - 2x_1 - 4 \Rightarrow -2x_2 + 2x_1 = 2x_2 - 2x_1 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2. \) Therefore \( f(x) \) is injective.

ii. \( y = \frac{x-2}{x+2} \Rightarrow y(x + 2) = x - 2 \Rightarrow yx - x + 2y = -2 \Rightarrow x(y - 1) + 2y = -2 \Rightarrow x(y - 1) = -2(1+y) \Rightarrow x = \frac{-2(1+y)}{y-1}. \) Given that \( 1 \notin B, \) taking \( f\left(\frac{-2(1+y)}{y-1}\right) = y. \) Therefore \( f(x) \) is surjective.

(b) Is the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x^3 \) (i) injective (ii) surjective and (iii) bijective? What if \( f : \mathbb{Q} \rightarrow \mathbb{Q} \) or \( f : \{-1, 0, 2\} \rightarrow \{-1, 0, 8\}? \) Prove or disprove.

\( \Rightarrow f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2. \)

\( \Rightarrow f^{-1}(x) = x^{1/3}. \)

We can now verify for each domain and codomain below:

i. \( f : \mathbb{R} \rightarrow \mathbb{R}. \)

Bijective. Both derivations above hold.
ii. \( f : \mathbb{Q} \rightarrow \mathbb{Q} \).
Injective, but not surjective. The cubed root is not always rational. For example, \( \sqrt[3]{2} \notin \mathbb{Q} \).

iii. \( f : \{-1, 0, 2\} \rightarrow \{-1, 0, 8\} \).
Bijective. Both derivations above hold.

7. Construct a bijection \( f : [a,b) \rightarrow [0,1) \). Show that \( f \) is one-to-one and onto.

\[
f(x) = \frac{x-a}{b-a}
\]

We need \( 0 \leq a < b \)

(a) It is straightforward to verify that \( f(x_1) = f(x_2) \Rightarrow \frac{x_1-a}{b-a} = \frac{x_2-a}{b-a} \Rightarrow (x_1-a)(b-a) = (x_2-a)(b-a) \Rightarrow x_1-a = x_2-a \Rightarrow x_1 = x_2 \). Therefore \( f(x) \) is one-to-one.

(b) \( y = \frac{x-a}{b-a} \Rightarrow y(b-a) = (x-a) \Rightarrow x = y(b-a) + a \Rightarrow f(x) = \frac{x(b-a)+a-a}{b-a} = x \Rightarrow f(x) \) is onto.