CS 4733 Class Notes: Kinematic Singularities and Jacobians

1 Kinematic Singularities

1. If we try to control a manipulaotr in Cartesian space, we can sometimes run into difficulties since the inverse mapping from Cartesian space to joint space can sometimes become a problem. These problem positions of the robot are referred to as singularities or degeneracies.

2. At a singularity, the mobility of a manipulator is reduced. Usually, arbitrary motion of the manipulator in a Cartesian direction is lost. This is referred to as "Losing a DOF".

3. Boundary Singularities (also known as workspace singularities) are a common type of singularity. They are usually caused by a full extension of a joint, and asking the manipulator to move beyond where it can be positioned. Typically, this is trying to reach out of the workspace at the farthest extent of the workspace.

4. Internal Singularities (also known as joint space singularities). They are generally caused by an alignment of the robots axes in space. For example, if 2 axes become aligned in space, rotation of one can be canceled by counterrotation of the other, leaving the actual joint location indeterminate. Also, certain kinematic alignments specific to each manipulator can cause these.

5. At a joint space singularity, infinite inverse kinematic solutions may exist.

6. At a joint space singularity, small Cartesian motions may require infinite joint velocities, causing a problem.

7. By analyzing the Jacobian matrix of a manipulator we can find the singular posiitons of the robot.

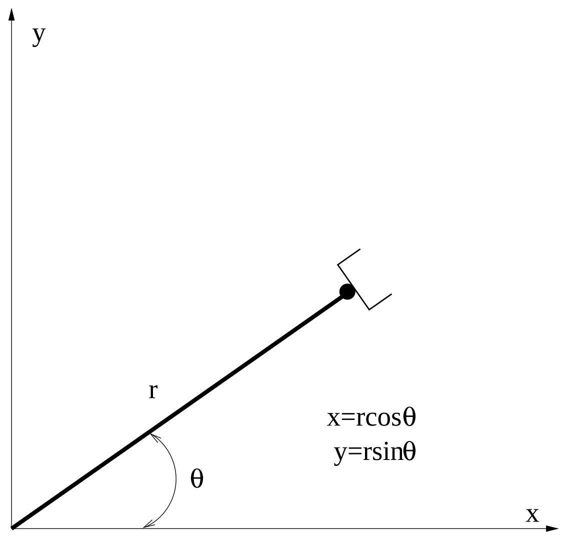

Figure 1: 2-DOF Polar, planar manipulator. The endpoint has coordinates $(rCos\theta\ ,\ rSin\theta)$.

2 Manipulator Jacobians

Because we control the robot in joint space and tend to reason about motion in Cartesian space, we need to fully understand the mapping from joint space to Cartesian space and vice-versa. Forward and inverse kinematics describe the static relationship between these spaces, but we must also understand the differential relationships.

To do this, we will define a mapping between small (differential) changes in joint space and how they create small (differential) changes in Cartesian space.

Figure 1 is a 2-DOF polar manipulator. Joint 1 is a revolute joint and joint 2 is a prismatic joint, with an endpoint of $(rCos\theta\ ,\ rSin\theta)$. Let us find the rate of change of x and y, i.e. their velocities, using the chain rule to differentiate x and y with respect to time t: [1]

$$\frac{dx}{dt} = \frac{\partial(r\ cos\theta)}{\partial r}\frac{dr}{dt} + \frac{\partial(r\ cos\theta)}{\partial\theta}\frac{d\theta}{dt} \quad ==> \quad \dot{x} = cos\theta\dot{r} - rsin\theta\dot{\theta}$$

$$\frac{dy}{dt} = \frac{\partial(r\ sin\theta)}{\partial r}\frac{dr}{dt} + \frac{\partial(r\ sin\theta)}{\partial\theta}\frac{d\theta}{dt} \quad ==> \quad \dot{y} = sin\theta\dot{r} + r\ cos\theta\dot{\theta}$$

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{d\theta}{dt} \\ \frac{dr}{dt} \end{bmatrix}$$

[1] Chain Rule:

$$z = F(x,y) \quad dz = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \frac{dz}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\, sin\,\theta & cos\,\theta \\ r\, cos\,\theta & sin\,\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix}$$

The matrix which relates changes in joint parameter velocities to Cartesian velocities is called the Jacobian Matrix. This is a time-varying, postion dependent linear transform. It has a number of columns equal to the number of degrees of freedom in joint space, and a number of rows equal to the number of degrees of freedom in Cartesian space.The Jacobian for this manipulator is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix} \quad where \quad J = \begin{bmatrix} -r\, sin\,\theta & cos\,\theta \\ r\, cos\,\theta & sin\,\theta \end{bmatrix}$$

If we specify the Cartesian velocities, we can find the joint parameter velocities with the inverse Jacobian. The inverse Jacobian is:

$$\left[J^{-1}\right] \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix} \quad where \quad J^{-1} = \begin{bmatrix} \frac{-sin\,\theta}{r} & \frac{cos\,\theta}{r} \\ cos\,\theta & sin\,\theta \end{bmatrix}$$

A singularity occurs when the joint velocity in joint space becomes infinite to maintain Cartesian velocity. It shows us where the continuity in joint space breaks down as related to Cartesian space. A singularity occurs whenever the determinant of the Jacobian is 0 (meaning we cannot invert it). The associated Jacobian matrix is said to be singular. To find when this occurs we set

$$det\,(J) = 0$$

and solve for the singularity. In this case, $det\,(J) = -r$. The determinant is 0 when $r = 0$. Since r is the radius of the manipulator, the robot has a *singularity* when we try to move through the origin in Cartesian space. At this point, the joint space velocity of joint 1 becomes infinite to achieve any Cartesian velocity vector(see figure 2).

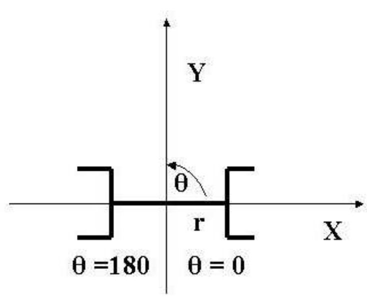

Figure 2: 2-Link Polar Manipulator near origin. If we establish a manipulator path that takes the gripper along the positive X axis to the negative X axis by decreasing r, we can see that joint 1 (θ) will have to rotate from 0 degrees to 180 degrees as the gripper passes through the origin. This rotation will cause the joint to have an infinite velocity as the configuration changes from the positive to negative X axis.

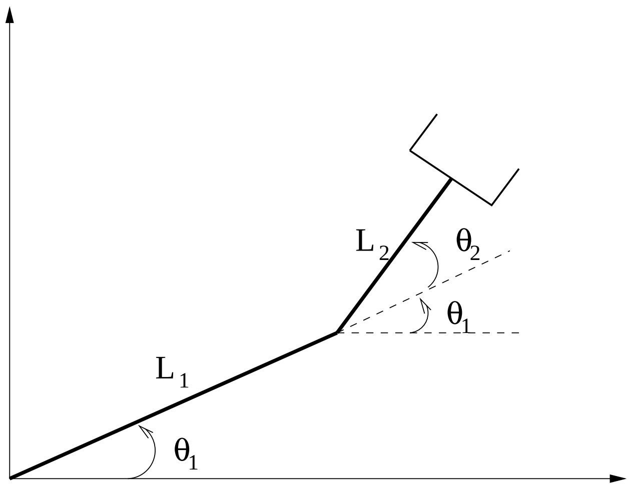

Figure 3: 2-Link RR Planar Manipulator

3 Jacobian of 2-link Revolute-Revolute (RR) Manipulator

$$T_1^0 = \begin{bmatrix} C_1 & -S_1 & 0 & C_1L_1 \\ S_1 & C_1 & 0 & S_1L_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^1 = \begin{bmatrix} C_2 & -S_2 & 0 & C_2L_2 \\ S_2 & C_2 & 0 & S_2L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^0 = \begin{bmatrix} C_{12} & -S_{12} & 0 & C_1L_1 + L_2C_{12} \\ S_{12} & C_{12} & 0 & S_1L_1 + L_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The T matrices for the manipulator are above (see figure 3. Since this is planar manipulator, we will find J such that

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = J \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \end{bmatrix} \quad ; \qquad J = \begin{bmatrix} \frac{\partial X}{\partial \Theta_1} & \frac{\partial X}{\partial \Theta_2} \\ \frac{\partial Y}{\partial \Theta_1} & \frac{\partial Y}{\partial \Theta_2} \end{bmatrix}$$

$$X = C_1L_1 + L_2C_{12} \; ; \; \frac{\partial X}{\partial t} = \frac{\partial(C_1L_1 + L_2C_{12})}{\partial \Theta_1}\frac{\partial \Theta_1}{\partial t} + \frac{\partial(C_1L_l + L_2C_{12})}{\partial \Theta_2}\frac{\partial \Theta_2}{\partial t}$$

$$\dot{X} = (-S_1L_1 - L_2S_{12})\dot{\Theta}_1 - L_2S_{12}\dot{\Theta}_2$$

$$Y = S_1L_1 + L_2S_{12} \; ; \; \frac{\partial Y}{\partial t} = \frac{\partial(S_1L_1 + L_2S_{12})}{\partial \Theta_1}\frac{\partial \Theta_1}{\partial t} + \frac{\partial(S_1L_1 + L_2S_{12})}{\partial \Theta_2}\frac{\partial \Theta_2}{\partial t}$$

$$\dot{Y} = (C_1L_1 + L_2C_{12})\dot{\Theta}_1 + L_2C_{12}\dot{\Theta}_2$$

$$and \quad \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} -S_1L_1 - L_2S_{12} & -L_2S_{12} \\ C_1L_1 + L_2C_{12} & L_2C_{12} \end{bmatrix} \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \end{bmatrix}$$

Again, the Jacobian matrix relates rate of change of joint variables to rate of change of Cartesian variables.

4 Finding Singularites of the 2-Link Manipulator

If we invert the Jacobian, we get:

$$J^{-1}\dot{X} = \dot{\Theta}$$

The inverse is undefined whenever det (J)=0. (It is a singular matrix.) So, by solving det (J)=0, we can find singularities in the robot workspace.

$$J = \begin{bmatrix} -S_1L_1 - L_2S_{12} & -L_2S_{12} \\ C_1L_1 + L_2C_{12} & L_2C_{12} \end{bmatrix}$$

$$\begin{aligned} det\,(J) &= (-S_1L_1 - L_2S_{12})(L_2C_{12}) + (L_2S_{12})(C_1L_1 + L_2C_{12}) \\ &= -S_1L_1L_2C_{12} - L_2^2S_{12}C_{12} + C_1L_1L_2S_{12} + L_2^2S_{12}C_{12} \\ &= L_1L_2(S_{12}C_1 - C_{12}S_1) = L_1L_2(S(\Theta_1 + \Theta_2 - \Theta_1)) \\ & det\,(J) \quad = \quad L_1L_2S\Theta_2 \end{aligned}$$

Setting this equal to zero, we find singular positions:

$$L_1L_2S\Theta_2 = 0$$

1. if L_1=0, cannot move arm radially; also Θ_1 indeterminate
2. if L_2=0, cannot move arm radially; also Θ_2 indeterminate
3. if S_2=0, arm is at full extension (θ_2=0), or looped back onto link 1 (θ_2=180)- again, we cannot move radially in Cartesian space. Loss of Cartesian motion.

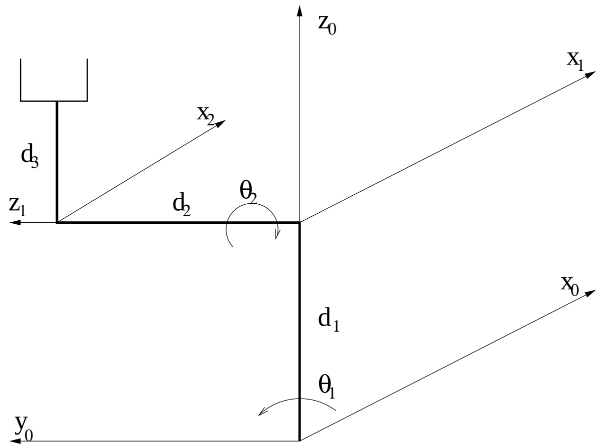

Figure 4: Stanford Arm Frame Diagram - first 3 joints only

5 Stanford Arm Jacobian - first 3 joints only

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} C_1 S_2 d_3 \,-\, S_1 d_2 \\ S_1 S_2 d_3 \,+\, C_1 d_2 \\ C_2 d_3 \,+\, d_1 \end{bmatrix} ; \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = J \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{d}_3 \end{bmatrix} \;;\; \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial \Theta_1} & \frac{\partial X}{\partial \Theta_2} & \frac{\partial X}{\partial d_3} \\ \frac{\partial Y}{\partial \Theta_1} & \frac{\partial Y}{\partial \Theta_2} & \frac{\partial Y}{\partial d_3} \\ \frac{\partial Z}{\partial \Theta_1} & \frac{\partial Z}{\partial \Theta_2} & \frac{\partial Z}{\partial d_3} \end{bmatrix} \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$J = \begin{bmatrix} -S_1 S_2 d_3 - C_1 d_2 & C_1 C_2 d_3 & C_1 S_2 \\ C_1 S_2 d_3 - S_1 d_2 & S_1 C_2 d_3 & S_1 S_2 \\ 0 & -S_2 d_3 & C_2 \end{bmatrix}$$

Singularities: when det (J)=0

$$\begin{aligned} det\,(J) &= S_2 d_3 (S_1 S_2 (-S_1 S_2 d_3 - C_1 d_2) - C_1 S_2 (C_1 S_2 d_3 - S_1 d_2)) \\ &\quad + C_2 ((-S_1 S_2 d_3 - C_1 d_2)(S_1 C_2 d_3) - (C_1 S_2 d_3 - S_1 d_2)(C_1 C_2 d_3)) \\ &= S_2 d_3 [-S_1^2 S_2^2 d_3 - S_1 S_2 C_1 d_2 - C_1^2 S_1^2 d_3 + C_1 S_1 S_2 d_2] \\ &\quad + C_2 [-S_1^2 S_2 C_2 d_3^2 - C_1 S_1 C_2 d_2 d_3 - C_1^2 S_2 C_2 d_3^2 + C_1 S_1 C_2 d_2 d_3] \\ &= S_2 d_3 [-d_3 S_2^2 [C_1^2 + S_1^2] + C_2 [-S_2 C_2 d_3^2 [S_1^2 + C_1^2]] \\ &= S_2 d_3 [-d_3 S_2^2] - C_2^2 S_2 d_3^2 \;=\; -S_2^3 d_3^2 - C_2^2 S_2 d_3^2 \;=\; -S_2 d_3^2 [S_2^2 + C_2^2] \\ det\,(J) &= -S_2 d_3^2 \end{aligned}$$

When d_3 = 0, the arm cannot move in Cartesian Z direction

When s_2 = 0, the arm is tangent to the workspace inner boundary; it cannot move along shoulder axis direction

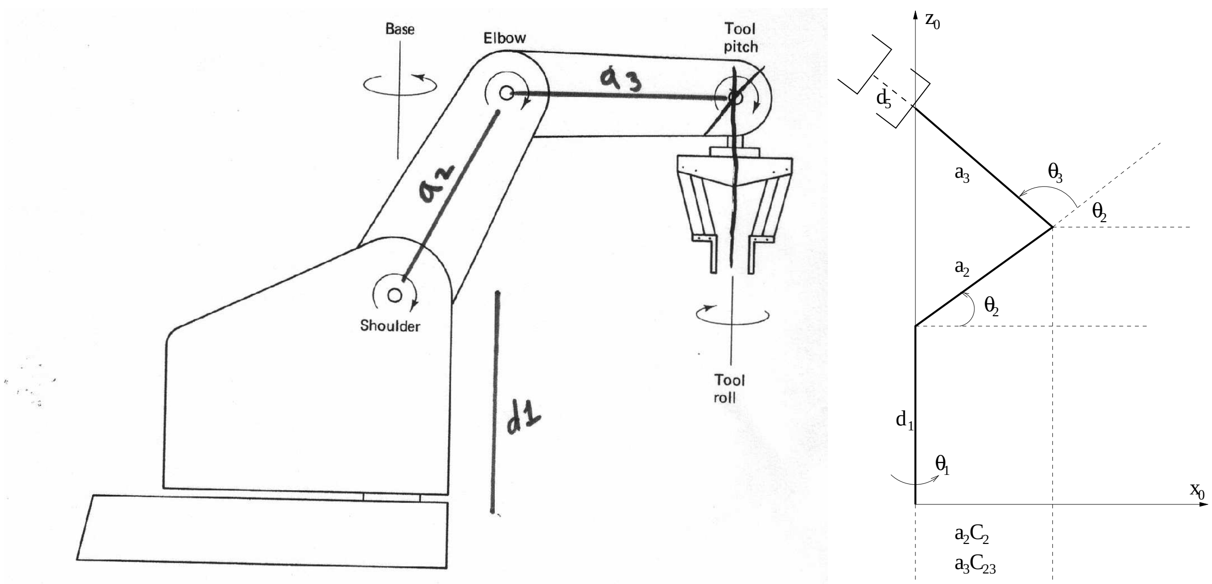

Figure 5: Left: Microbot Robot. Right: Singular position for reduced (i.e 3 axis - ignore tool roll joint) Microbot Robot

6 Microbot Robot Jacobian - First 3 joints

The solution ot the first three joints of the Microbot Robot is:

Joint	θ	d	a	α
1	θ_1	d_1	0	-90
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0

$$T_3^0 = \begin{bmatrix} | & | & | & c_1(a_2c_2 + a_3c_{23}) \\ N & S & A & s_1(a_2c_2 + a_3c_{23}) \\ | & | & | & d_1 - a_2s_2 - a_3s_{23} \\ | & | & | & 1 \end{bmatrix}$$

Find Jacobian J such that

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = J \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \end{bmatrix} \; ; \; J = \begin{bmatrix} \frac{\partial X}{\partial \Theta_1} & \frac{\partial X}{\partial \Theta_2} & \frac{\partial X}{\partial \Theta_3} \\ \frac{\partial Y}{\partial \Theta_1} & \frac{\partial Y}{\partial \Theta_2} & \frac{\partial Y}{\partial \Theta_3} \\ \frac{\partial Z}{\partial \Theta_1} & \frac{\partial Z}{\partial \Theta_2} & \frac{\partial Z}{\partial \Theta_3} \end{bmatrix} \qquad \begin{matrix} \dot{X} = J\dot{\Theta} \\ J^{-1}\dot{X} = \dot{\Theta} \end{matrix}$$

$$J = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -s_2c_1a_2 - s_{23}a_3c_1 & -s_{23}a_3c_1 \\ c_1(a_2c_2 + a_3c_{23}) & -s_2s_1a_2 - s_{23}a_3s_1 & -s_{23}a_3s_1 \\ \underbrace{0}_{joint\,1\,has\,no\,effect\,on\,Cartesian\,Z\,velocity} & -c_2a_2 - c_{23}a_3 & -c_{23}a_3 \end{bmatrix}$$

To find singularities, we set Jacobian Det = 0

$$det(J) =$$
$$(c_2a_2 + a_3c_{23})[-s_1(a_2c_2 + a_3c_{23})(-s_{23}a_3s_1) + c_1(a_2c_2 + a_3c_{23})(s_{23}a_3c_1)]$$
$$-c_{23}a_3[s_1(a_2c_2 + a_3c_{23})(s_2s_1a_2 + s_{23}a_3s_1) + c_1(a_2c_2 + a_3c_{23})(s_2c_1a_2 + s_{23}a_3c_1)]$$

$$\begin{aligned}
&= (c_2a_2 + c_{23}a_3)[s_1^2a_2a_3c_2s_{23} + s_1^2a_3^2s_{23}c_{23} + c_1^2a_3a_2c_2s_{23} + c_1^2a_3^2c_{23}s_{23}] \\
&\quad -c_{23}a_3[s_1^2s_2a_2^2c_2 + s_1^2s_2a_2a_3c_{23} + s_1^2a_2c_2s_{23}a_3 + s_1^2a_3^2s_{23}c_{23} \\
&\quad\quad +c_1^2a_2^2c_2s_2 + c_1^2s_2a_2a_3c_{23} + c_1^2s_{23}a_3a_2c_2 + c_1^2a_3^2s_{23}c_{23}] \\
&= [c_2a_2 + c_{23}a_3][a_2a_3c_2s_{23} + a_3^2s_{23}c_{23}] \\
&\quad -c_{23}a_3[a_2^2c_2s_2 + s_2a_2a_3c_{23} + s_{23}a_3a_2c_2 + a_3^2s_{23}c_{23}]
\end{aligned}$$

$$\begin{aligned}
&= [c_2a_2 + c_{23}a_3][a_3s_{23}(a_2c_2 + a_3c_{23})] \\
&\quad -c_{23}a_3[a_2s_2(a_2c_2 + a_3c_{23}) + a_3s_{23}(a_2c_2 + a_3c_{23})] \\
&= [c_2a_2 + c_{23}a_3][a_3s_{23}][a_2c_2 + a_3c_{23}] \\
&\quad -c_{23}a_3[(a_2s_2 + a_3s_{23})(a_2c_2 + a_3c_{23})] \\
&= [a_2c_2 + a_3c_{23}][(a_3s_{23})(a_2c_2 + a_3c_{23}) - c_{23}a_3(a_2s_2 + a_3c_{23})] \\
&= [a_2c_2 + a_3c_{23}][a_3s_{23}a_2c_2 - c_{23}a_3a_2s_2] \\
&= [a_2c_2 + a_3c_{23}][a_3a_2(\underbrace{s_{23}c_2 - c_{23}s_2}_{s(\Theta_2+\Theta_3-\Theta_2)=s\Theta_3})] \\
&= \underline{[a_2c_2 + a_3c_{23}][a_3a_2s_3]} \ \surd
\end{aligned}$$

Analysis: det (J) = 0 when

1. $a_3 = 0$; Lose link length
2. $a_2 = 0$; Lose link length
3. $s_3 = 0$; Elbow forearm fully extended
4. $[a_2c_2 + a_3c_{23}] = 0$; This is the projection of the forearem and upper arm onto the x axis. If they sum to 0, then arm is over the origin, and joint 1 loses its ability to position the robot (see figure 5.

7 How to Compute a Matrix Inverse

NOTE: This method is only efficient for small matrices!

Given

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

its Determinant, det A = $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

where A_{ij} is the determinant of the minor matrix formed by deleting the row i and column j.

$$A_{ij} = (-1)^{i+j} det\, M_{ij}$$

A_{ij} is the cofactor.

$$So\ det\ A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Now, look at the matrix product:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \underline{\begin{bmatrix} det\,A & 0 & 0 \\ 0 & det\,A & 0 \\ 0 & 0 & det\,A \end{bmatrix}}$$

each diagonal entry is the det A, and off diagonal entries are zero (Try it, you will see they vanish). We can write the above product as:

$$A \cdot A^T_{cofactors} = (det\,A)I$$

$$A \cdot \frac{A^T_{cofactors}}{det\,(A)} = I$$

and inverse of A is $\frac{A^T_{cofactors}}{det\,(A)} = A^{-1}$

Note the $A^T_{cofactors}$ is a transpose of the minors of A appended with correct sign.

Example: Find Inverse of

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 6 \\ 4 & 0 & 5 \end{bmatrix} \qquad A^T_{cofactors} = \begin{bmatrix} 10 & -15 & 18 \\ 24 & 5 & -6 \\ -8 & 12 & 2 \end{bmatrix}$$

det A = 82

$$A^{-1} = \frac{A^T_{cofactors}}{det\,(A)} = \underbrace{\begin{bmatrix} 10/82 & -15/82 & 18/82 \\ 24/82 & 5/82 & -6/82 \\ -8/82 & 12/82 & 2/82 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 6 \\ 4 & 0 & 5 \end{bmatrix}}_{Multiply\ these\ two = I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$