Tractability of multivariate approximation over weighted standard Sobolev spaces

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Abstract

We study multivariate approximation for complex-valued functions defined over the d-dimensional torus, these functions belonging to weighted standard Sobolev spaces of smoothness r. Algorithms are allowed to use finitely many arbitrary continuous linear functionals, which may be chosen adaptively. The error of an algorithm is measured by the L_2 norm, in the the worst case setting. No matter how we choose positive weights, we prove that multivariate approximation cannot be quasi-polynomially tractable, meaning that the minimal number of continuous linear functionals needed to find an ε-approximation in the d-variate case grows faster than any polynomial in $\varepsilon^{-\ln d}$. We also study (s,t)-weak tractability for positive s and t, meaning that the minimal number of continuous linear functionals is *not* exponential in ε^{-t} and d^s. We restrict our attention to product weights, defined as products of powers of weightlets γ_j. We obtain conditions on the weightlets that are necessary and sufficient for our problem to be (s,t)-weakly tractable. In particular, if $rs < 2$ and $t \le 1$, then our problem is (s,t)-weakly tractable iff $\gamma_j = o(j^{-(2-rs)/(rs)})$ as $j \to \infty$.

Keywords: tractability, information-based complexity, multivariate approximation, weighted Sobolev space
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1. Introduction

Over the last quarter-century, research in information-based complexity theory has focused on tractability studies. The level of tractability of a specific problem defined over high-dimensional domains reflects the extent to which that problem can be solved with reasonable cost. Rather than studying problems defined for classical (isotropic) Sobolev spaces, tractability research has mainly dealt with function spaces having a (non-isotropic) tensor product structure. Most typically, this has meant looking at functions over the unit d-cube I^d (where d is large), constructed as a d-fold tensor product of functions defined over the unit interval $I = [0,1]$. Many results for such problems are found in the monograph series [1, 2, 3].

In 2014, Kühn, Sickel and Ullrich [4] took a new tack, studying the tractability of multivariate approximation over standard isotropic Sobolev spaces $H^r(\mathbb{T}^d)$ for complex valued-functions defined over the d-dimensional torus $\mathbb{T}^d$. The problem elements were chosen to be the unit ball of $H^r(\mathbb{T}^d)$, but under three different norms; these norms were equivalent for any given d, with equivalence constants depending on d. They consider algorithms that adaptively use finitely many linear functionals from the class Λ^{all} of information, consisting of all continuous linear functionals.

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Measuring the error of an algorithm in the $L_2(\mathbb{T}^d)$-norm and in the worst case setting, they studied the information complexity $n(\varepsilon, d)$, defined as the minimal number of continuous linear functionals needed to find an approximation whose error is at most ε in the d-variate case, where $\varepsilon \in (0, 1)$.

The reason for choosing $\mathbb{T}^d$ instead of I^d is one of practicality. It is well-known that $n(\varepsilon, d)$ is the smallest integer n such that the $(n+1)$st eigenvalue of a particular operator is at most ε^2. For the (non-isotropic) tensor product cases that had been extensively studied in the past, we can explicitly determine these eigenvalues when the functions are defined over I^d. But when we turn to classical isotropic Sobolev spaces, we simply don't know these eigenvalues for the d-cube, but we do know them for the d-torus. So we now use tori rather than cubes when studying tractability for classical Sobolev spaces. We hope that the eigenvalues for the d-cube will be found in the future, which would allow us to study tractability also over the d-cube.

Another obstacle in studying the isotropic case even for the d-dimensional torus is that its eigenvalues have a different form than those for the multivariate tensor product problems that had been previously studied. Whereas the eigenvalues for multivariate tensor product problems are given by products of the eigenvalues for univariate problems, the eigenvalues for multivariate approximation over standard Sobolev spaces are given by sums. Hence determining tractability of multivariate approximation over standard Sobolev spaces requires different proof techniques than those used for the tensor product case.

For each of the three norms mentioned above in [4], Kühn *et al.* found a condition that was necessary for weak tractability, as well as one that was sufficient. However, there was a gap between these two conditions. Several papers were able to fill this gap, albeit via taking different directions:

- Siedlecki and Weimar [5] introduced the concept of (s, t)-*weak tractability*, which generalizes weak tractability, since weak tractability is $(1, 1)$-weak tractability. They were able to determine the values of s and t for which the $L_2(\mathbb{T}^d)$-approximation of $H^r(\mathbb{T}^d)$ is (s, t) weakly tractable, and in so doing, were the first to fill the gap mentioned above.

- Kühn, Mayer, and Ullrich [6] used entropy numbers, and characterized weak tractability for a family of norms on $H^r(\mathbb{T}^d)$ that includes the norms of [4] as special cases.

- Chen and Wang [7] studied *anisotropic spaces*, including isotropic spaces as a special case.

- The current authors [8] gave a new general characterization of (s, t)-weak tractability that did not require reordering the eigenvalues mentioned above. They used this new technique to successfully determine the (s, t)-weak tractability of the $L_2(\mathbb{T}^d)$-approximation problem for $H^r(\mathbb{T}^d)$.

The paper of Kühn, Sickel and Ullrich was for *unweighted* isotropic Sobolev spaces. One of the norms studied was the standard Sobolev norm, whose square is the sum of all derivatives $\|D^{\mathbf{m}} f\|_{L_2}^2$ of order $|\mathbf{m}| := m_1 + m_2 + \cdots + m_d \le r$. In the present paper we consider *weighted* Sobolev spaces, where the square of the norm is the sum of all weighted derivatives $\gamma_{d,\mathbf{m}}^{-1} \|D^{\mathbf{m}} f\|_{L_2}^2$ of order $|\mathbf{m}| \le r$, where $\gamma_{d,\mathbf{m}} \in (0, 1]$. Hence, the unit ball can shrink a great amount when going from the unweighted case to the weighted case. This might lead us to guess that some of the negative tractability results for the unweighted case might become positive when we go to the weighted case. As we shall see, this is only partially true.

By analogy with the tensor product case, one may hope that if the weights $\gamma_{d,\mathbf{m}}$ decay quickly enough, our problem might be one of the following:

- *Strongly polynomially tractable* (SPT), meaning that the information complexity can be bounded by a polynomial in ε^{-1} independently of d.

- *Polynomially tractable* (PT), meaning that the information complexity can be bounded by a polynomial in ε^{-1} and d.

- *Quasi-polynomially tractable* (QPT), meaning that the information complexity is bounded by a power of $\varepsilon^{-\ln d}$ for large d.

Obviously SPT $\implies$ PT $\implies$ QPT.

Our first result is quite negative, perhaps even disheartening: no matter how we choose the positive weights $\gamma_{d,\mathbf{m}}$, the multivariate approximation problem over weighted standard Sobolev spaces is *not* QPT. That is, the information complexity tends to infinity faster than any power of $\varepsilon^{-\ln d}$ independently of the weights.

So the only way we can get a positive tractability result is if we are willing to settle for a level of tractability that is even weaker than QPT. One such notion was given in [5]: For positive s and t, we say that our problem is (s,t)-*weakly tractable* (WT) if the information complexity is not exponential in ε^{-s} and d^t. It turns out that (s,t)-WT strongly depends on how s and t are related to the smoothness parameter r.

We first suppose that $rs > 2$ or $t > 1$. Then (s,t)-WT holds for *all* weights, including the (most difficult) unweighted case $\gamma_{d,\mathbf{m}} = 1$. This result was proved for some special cases in [4], with the more general case being established in [5]. This result was extended to include periodic Sobolev spaces with hybrid smoothness in [9].

So it only remains to consider the case in which $rs \le 2$ and $t \in (0,1]$. Moreover, we further restrict our attention to *product weights* having the form

$$\gamma_{d,\mathbf{m}} = \prod_{j=1}^{d} \gamma_j^{m_j} \qquad \text{with} \quad 0 < \gamma_d \le \gamma_{d-1} \le \cdots \le \gamma_1 \le 1,$$

where the *weightlets* $\gamma_1, \gamma_2, \ldots, \gamma_d$ are bounded and dimension-independent. We obtain the following results:

- Let $rs = 2$ and $t \in (0,1]$. Then our approximation problem is (s,t)-WT iff
$$\gamma_j = \begin{cases} o(1) & \text{for } t = 1, \\ o\left((\ln j)^{-1}\right) & \text{for } t \in (0,1), \end{cases} \qquad \text{as } j \to \infty.$$
- Let $rs < 2$ and $t \in (0,1]$. Then our approximation problem is (s,t)-WT iff
$$\gamma_j = o(j^{-(2-rs)/(rs)}) \qquad \text{as } j \to \infty.$$

Note the following:

1. The weight conditions do not hold for the unweighted case $\gamma_j \equiv 1$. This gives us another proof that the unweighted case is not (s,t)-WT for any positive s and t such that $rs \le 2$ and $t \in (0,1]$.
2. The dependence on t is only present when $rs = 2$;
   (a) When $t = 1$, we only require that the weights go to zero, their speed of convergence being irrelevant.
   (b) When $t < 1$ then the weights must go to zero faster than $1/\ln\, j$.
3. The situation is quite different when $rs < 2$. Now the γ_j must then go to zero faster than a polynomial in j^{-1} whose exponent depends on rs. For small rs, this exponent is large and goes to infinity as rs goes to zero. Moreover, the speed at which γ_j converges is independent of t.

From the results given above, it is easy to obtain weight conditions that are necessary and sufficient for *uniform weak tractability*, meaning that (s,t)-WT holds for all positive s and t, as initially studied in [10]. Our problem is uniformly weakly tractable iff

$$\lim_{j\to\infty} \gamma_j j^p = 0 \qquad \text{for all positive } p,$$

or (equivalently) iff

$$\lim_{j\to\infty} \frac{\ln\, \gamma_j^{-1}}{\ln\, j} = \infty.$$

We end the introduction by briefly discussing the class Λ^{std} of information, which consists of only function evaluations. It is well-known that the functional $L_x(f) = f(x)$ for all $f \in H^r(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$ is continuous iff $r > d/2$. Since we consider a fixed smoothness parameter r and varying d, the last condition does not hold for large d. In this case, L_x is not even well-defined. Of course, we can switch to the space $H^r(\mathbb{T}^d) \cap C(\mathbb{T}^d)$, where $C(\mathbb{T}^d)$ is a space of continuous functions. Then L_x is well defined but still is not continuous for $d \ge 2r$. It is proven in [11] that discontinuous linear functionals are useless for approximation of continuous operators. From this result it follows that for the class Λ^{std} the minimal worst case errors of all algorithms using n function evaluations is equal to the norm of the continuous operator we consider. In our case, for multivariate approximation, this norm is one. This means that the class Λ^{std} can be reasonable used for multivariate approximation over the space $H^r(\mathbb{T}^d)$ only if $r = r(d)$ varies with d and $r(d) > d/2$. This is left as a future research project.

2. Problem description

We shall be using standard notation. In particular, we let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$ denote the strictly positive integers, the non-negative integers, the reals and the complex numbers respectively. We use boldface roman and italic letters to denote vectors with integer or real components, respectively.

For $d \in \mathbb{N}$ and for a multi-index $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$, we write $|\mathbf{k}| = k_1 + \cdots + k_d$. We also write $\boldsymbol{x}^{\mathbf{k}} = \prod_{j=1}^d x_j^{k_j}$ and $\mathbf{k} \cdot \boldsymbol{x} = \sum_{j=1}^d k_j x_j$ for $\boldsymbol{x} \in \mathbb{R}^d$ and $\mathbf{k} \in \mathbb{N}_0^d$. In addition, we write $\|\cdot\|_{\ell_2(\mathbb{R}^d)}$ for the usual vector norm

$$\|\boldsymbol{x}\|_{\ell_2(\mathbb{R}^d)} = \left(\sum_{j=1}^d x_j^2\right)^{1/2} \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^d.$$

Let $\mathbb{T}^d = [0, 2\pi]^d$ denote the d-torus, identifying opposite points, so that $f\colon \mathbb{T}^d \to \mathbb{C}$ satisfies

$$f(\boldsymbol{x}) = f(\boldsymbol{y}) \qquad \text{whenever } \boldsymbol{x} - \boldsymbol{y} \in 2\pi\{-1, 0, 1\}^d.$$

Note that functions defined over $\mathbb{T}^d$ are periodic, with period 2π. We let $L_2(\mathbb{T}^d)$ denote square-integrable complex-valued functions defined over $\mathbb{T}^d$, with inner product

$$\langle f, g\rangle_{L_2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} \qquad \forall\, f, g \in L_2(\mathbb{T}^d).$$

Let

$$e_{d,\mathbf{k}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}} \exp(\mathrm{i}\,\mathbf{k} \cdot \boldsymbol{x}) \qquad \forall\, \mathbf{k} \in \mathbb{Z}^d, d \in \mathbb{N}. \tag{2.1}$$

Then $\{\, e_{d,\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d \,\}$ is an orthonormal basis for $L_2(\mathbb{T}^d)$, and we have

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{d,\mathbf{k}}(f) e_{d,\mathbf{k}} \qquad \forall\, f \in L_2(\mathbb{T}^d),$$

with

$$c_{d,\mathbf{k}} = \langle f, e_{d,\mathbf{k}}\rangle_{L_2(\mathbb{T}^d)} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(\boldsymbol{x}) \exp(-\mathrm{i}\,\mathbf{k} \cdot \boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \qquad \forall\, f \in L_2(\mathbb{T}^d), \mathbf{k} \in \mathbb{Z}^d.$$

Moreover

$$\langle f, g\rangle_{L_2(\mathbb{T}^d)} = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{d,\mathbf{k}}(f)\overline{c_{d,\mathbf{k}}(g)} \qquad \forall\, f, g \in L_2(\mathbb{T}^d).$$

For $r \in \mathbb{N}_0$, let us define a family

$$\Gamma = \bigcup_{d\in\mathbb{N}} \left\{\, \gamma_{d,\mathbf{m}} \in (0, 1] : \mathbf{m} \in \mathbb{N}_0^d \text{ such that } |\mathbf{m}| \le r \right\}$$

of positive weights. We assume that $\gamma_{d,\mathbf{0}} = 1$. Let

$$H^r(\mathbb{T}^d) = \{\, f \in L_2(\mathbb{T}^d) : D^{\mathbf{m}} f \in L_2(\mathbb{T}^d) \text{ for all } \mathbf{m} \in \mathbb{Z}^d \text{ with } |\mathbf{m}| \le r \,\}$$

be the classical unweighted Sobolev space, under the standard inner product

$$\langle f, g\rangle_{H^r(\mathbb{T}^d)} = \sum_{|\mathbf{m}|\le r} \langle D^{\mathbf{m}} f, D^{\mathbf{m}} g\rangle_{L_2(\mathbb{T}^d)}, \qquad \forall\, f, g \in H^r(\mathbb{T}^d),$$

where $D^{\mathbf{m}} = \partial^{|\mathbf{m}|} / (\partial^{m_1} x_1 \cdots \partial^{m_d} x_d)$, as usual.

Then the *weighted Sobolev space* $H^r_\Gamma(\mathbb{T}^d)$ is defined to be $H^r(\mathbb{T}^d)$, under the inner product

$$\langle f, g\rangle_{H^r_\Gamma(\mathbb{T}^d)} = \sum_{|\mathbf{m}|\le r} \gamma_{d,\mathbf{m}}^{-1} \langle D^{\mathbf{m}} f, D^{\mathbf{m}} g\rangle_{L_2(\mathbb{T}^d)}.$$

Clearly, the norm $\|\cdot\|_{H^r_\Gamma(\mathbb{T}^d)}$ is equivalent to the classical Sobolev norm $\|\cdot\|_{H^r(\mathbb{T}^d)}$, with

$$\left(\min_{\mathbf{m}:\,|\mathbf{m}|\le r} \gamma_{d,\mathbf{m}}^{-1/2}\right) \|f\|_{H^r(\mathbb{T}^d)} \le \|f\|_{H^r_\Gamma(\mathbb{T}^d)} \le \left(\max_{\mathbf{m}:\,|\mathbf{m}|\le r} \gamma_{d,\mathbf{m}}^{-1/2}\right) \|f\|_{H^r(\mathbb{T}^d)},$$

and so $H^r_\Gamma(\mathbb{T}^d)$ is a Hilbert space.

We are ready to define the *multivariate approximation problem* as

$$\mathrm{App}_\Gamma = \{\, \mathrm{App}_{\Gamma,d} : d \in \mathbb{N}\,\},$$

where $\mathrm{App}_{\Gamma,d} \colon H^r_\Gamma(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ is given by the canonical embedding

$$\mathrm{App}_{\Gamma,d}\, f = f \qquad \forall\, f \in H^r_\Gamma(\mathbb{T}^d).$$

Clearly, $\|\, \mathrm{App}_{\Gamma,d}\, f\|_{L_2(\mathbb{T}^d)} \le \|f\|_{H^r_\Gamma(\mathbb{T}^d)}$, with equality for $f = 1$. This means that

$$\|\, \mathrm{App}_{\Gamma,d}\,\| = 1 \qquad \text{for all } d \text{ and all weights } \Gamma.$$

We briefly remind the reader about a few standard concepts and results of information-based complexity, see (e.g.) [1] or [12] for details. Our goal is to approximate

$$\mathrm{App}_{\Gamma,d}\, f = f \approx A_n(f) \qquad \text{for } f \text{ in the } H^r_\Gamma(\mathbb{T}^d)\text{-unit ball},$$

where A_n is an *algorithm* using *information* of cardinality at most n, i.e., $A_n(f)$ uses the values of at most n continuous linear functionals on f. Hence we may write

$$A_n(f) = \phi_n(L_1(f), L_2(f), \ldots, L_n(f)),$$

where $\phi_n \colon \mathbb{C}^n \to L_2(\mathbb{T}^d)$ and the functionals $L_1, \ldots, L_n \in [H^r_\Gamma(\mathbb{T}^d)]^*$ can all be chosen adaptively, the latter meaning that

$$L_j = L_j(\cdot, L_1(f), L_2(f), \ldots, L_{j-1}(f)) \qquad \text{for } j \in \{1, \ldots, n\}.$$

The cardinality n can also be chosen adaptively, see [1, 12].

We will measure error in the *worst case setting*, so that

$$e(A_n) = \sup_{\|f\|_{H^r_\Gamma(\mathbb{T}^d)}\le 1} \|\, \mathrm{App}_{\Gamma,d}\, f - A_n(f)\|_{L_2(\mathbb{T}^d)}.$$

Let $\lambda_{d,1} = 1 \ge \lambda_{d,2} \ge \cdots > 0$ be the eigenvalues of

$$W_{\Gamma,d} = \mathrm{App}^*_{\Gamma,d}\, \mathrm{App}_{\Gamma,d} \;:\; H^r_\Gamma(\mathbb{T}^d) \;\to\; H^r_\Gamma(\mathbb{T}^d),$$

with corresponding eigenvectors $e_{d,1}, e_{d,2}, \ldots$ that form an orthonormal basis for $H^r_\Gamma(\mathbb{T}^d)$. It is well-known that the algorithm

$$A^*_n(f) = \sum_{k=1}^{n} \langle f, e_{d,k}\rangle_{H^r_\Gamma(\mathbb{T}^d)} e_{d,k} \qquad \forall\, f \in H^r_\Gamma(\mathbb{T}^d)$$

has minimal error among all algorithms using information of cardinality at most n and

$$e(A^*_n) = \sqrt{\lambda_{d,n+1}}.$$

Finally, the *information complexity* $n(\varepsilon, \mathrm{App}_{\Gamma,d})$ of our d-variate approximation problem is the minimal n for which $e(A^*_n) \le \varepsilon$, where $\varepsilon \in (0,1)$. Hence we have

$$n(\varepsilon, \mathrm{App}_{\Gamma,d}) = \left|\left\{\, n \in \mathbb{N}_0 : \lambda_{d,n+1} > \varepsilon^2 \,\right\}\right| \qquad \forall\, \varepsilon \in (0,1). \tag{2.2}$$

In addition, the algorithm A^*_n with $n = n(\varepsilon, \mathrm{App}_{\Gamma,d})$ is *optimal*, in the sense that it provides an approximation having error at most ε, using information of minimal cardinality.

3. Spectral results

We now determine the eigenvalues and eigenvectors of $W_{\Gamma,d}$. By the results in the previous section, these results will allow us to explicitly determine the information complexity (and optimal algorithms) for our problem, which (in turn) will be needed for studying its tractability.

Lemma 3.1. *The set* $\{ e_{d,\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d \}$*, with* $e_{d,\mathbf{k}}$ *defined by* (2.1)*, is an orthogonal basis for* $H^r_\Gamma(\mathbb{T}^d)$*, with*

$$\|e_{d,\mathbf{k}}\|^2_{H^r_\Gamma(\mathbb{T}^d)} = \sum_{|\mathbf{m}|\le r} \gamma^{-1}_{d,\mathbf{m}} \mathbf{k}^{2\mathbf{m}}. \tag{3.1}$$

Proof. Since $D^{\mathbf{m}} e_{d,\mathbf{k}} = (\mathrm{i}\,\mathbf{k})^{\mathbf{m}} e_{d,\mathbf{k}}$ for any $\mathbf{k} \in \mathbb{Z}^d$ and $\mathbf{m} \in \mathbb{N}^d_0$, we may use periodicity to find that

$$\begin{aligned} \langle v, e_{d,\mathbf{k}}\rangle_{H^r_\Gamma(\mathbb{T}^d)} &= \sum_{|\mathbf{m}|\le r} \gamma^{-1}_{d,\mathbf{m}} \overline{(\mathrm{i}\,\mathbf{k})^{\mathbf{m}}} \langle D^{\mathbf{m}} v, e_{d,\mathbf{k}}\rangle_{L_2(\mathbb{T}^d)} \\ &= \sum_{|\mathbf{m}|\le r} \gamma^{-1}_{d,\mathbf{m}} (\mathrm{i}\,\mathbf{k})^{\mathbf{m}} \langle v, D^{\mathbf{m}} e_{d,\mathbf{k}}\rangle_{L_2(\mathbb{T}^d)} \qquad \forall\, \mathbf{k} \in \mathbb{Z}^d, v \in H^r_\Gamma(\mathbb{T}^d). \\ &= \Big(\sum_{|\mathbf{m}|\le r} \gamma^{-1}_{d,\mathbf{m}} \mathbf{k}^{2\mathbf{m}}\Big) \langle v, e_{d,\mathbf{k}}\rangle_{L_2(\mathbb{T}^d)} \end{aligned} \tag{3.2}$$

In particular,

$$\langle e_{d,\mathbf{p}}, e_{d,\mathbf{k}}\rangle_{H^r_\Gamma(\mathbb{T}^d)} = \Big(\sum_{|\mathbf{m}|\le r} \gamma^{-1}_{d,\mathbf{m}} \mathbf{k}^{2\mathbf{m}}\Big) \delta_{\mathbf{k},\mathbf{p}} \qquad \forall\, \mathbf{k}, \mathbf{p} \in \mathbb{Z}^d.$$

This shows that $\{ e_{d,\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d \}$ is orthogonal and (setting $\mathbf{p} = \mathbf{k}$) that (3.1) holds. To see that this set is also a basis, let $v \in H^r_\Gamma(\mathbb{T}^d)$ be such that $\langle v, e_{d,\mathbf{k}}\rangle_{H^r_\Gamma(\mathbb{T}^d)} = 0$ for all $\mathbf{k} \in \mathbb{Z}^d$. From (3.2), it follows that $\langle v, e_{d,\mathbf{k}}\rangle_{L_2(\mathbb{T}^d)} = 0$ for all $\mathbf{k} \in \mathbb{Z}^d$. Since $\{ e_{d,\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d \}$ is a basis for $L_2(\mathbb{T}^d)$, it follows that $v = 0$. Thus $\{ e_{d,\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d \}$ is a complete orthogonal set, as required. □

Lemma 3.2. *Let*

$$\lambda_{d,\mathbf{k}} = \|e_{d,\mathbf{k}}\|^{-2}_{H^r_\Gamma(\mathbb{T}^d)} = \Big(\sum_{|\mathbf{m}|\le r} \gamma^{-1}_{d,\mathbf{m}} \mathbf{k}^{2\mathbf{m}}\Big)^{-1} = \frac{1}{1 + \sum_{1\le|\mathbf{m}|\le r} \gamma^{-1}_{d,\mathbf{m}} \mathbf{k}^{2\mathbf{m}}}.$$

Then $\{ (e_{d,\mathbf{k}}, \lambda_{d,\mathbf{k}}) : \mathbf{k} \in \mathbb{Z}^d \}$ *is an eigensystem for* $W_{\Gamma,d}$.

Proof. For $\mathbf{k}, \mathbf{p} \in \mathbb{Z}^d$, we have

$$\langle W_{\Gamma,d} e_{d,\mathbf{k}}, e_{d,\mathbf{p}}\rangle_{H^r_\Gamma(\mathbb{T}^d)} = \langle \mathrm{App}_{\Gamma,d}\, e_{d,\mathbf{k}}, \mathrm{App}_{\Gamma,d}\, e_{d,\mathbf{k}}\rangle_{L_2(\mathbb{T}^d)} = \langle e_{d,\mathbf{k}}, e_{d,\mathbf{p}}\rangle_{L_2(\mathbb{T}^d)} = \delta_{\mathbf{k},\mathbf{p}}.$$

Since $\{ e_{d,\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d \}$ is a basis for $H^r_\Gamma(\mathbb{T}^d)$, it follows that $W_{\Gamma,d} e_{d,\mathbf{k}}$ must be a multiple of $e_{d,\mathbf{k}}$. Letting $\lambda_{d,\mathbf{k}}$ be that multiplier, we have

$$\lambda_{d,\mathbf{k}} \|e_{d,\mathbf{k}}\|^2_{H^r_\Gamma(\mathbb{T}^d)} = \langle W_{\Gamma,d} e_{d,\mathbf{k}}, e_{d,\mathbf{k}}\rangle_{H^r_\Gamma(\mathbb{T}^d)} = \langle e_{d,\mathbf{k}}, e_{d,\mathbf{k}}\rangle_{L_2(\mathbb{T}^d)} = 1,$$

and so $\lambda_{d,\mathbf{k}} = \|e_{d,\mathbf{k}}\|^{-2}_{H^r(\mathbb{T}^d)}$, as required. □

Suppose momentarily that $r = 0$. Then $H^r_\Gamma(\mathbb{T}^d) = L_2(\mathbb{T}^d)$ and all eigenvalues $\lambda_{d,\mathbf{k}} = 1$. This means that $n(\varepsilon, \mathrm{App}_{\Gamma,d}) = \infty$ for all $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$. Hence, the multivariate approximation problem App_Γ cannot be solved when $r = 0$. Hence in the rest of this paper, we shall assume that $r \ge 1$. Using (2.2) and Lemma 3.2, it follows that $n(\varepsilon, \mathrm{App}_{\Gamma,d}) < \infty$ for all $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$.

4. Quasi-polynomial intractability

The following lemma will help us study various kinds of tractability.

Lemma 4.1. *Let $\nu \in \mathbb{N}$, and suppose that $\psi \colon \mathbb{R}^\nu \to \mathbb{R}$ satisfies the following conditions:*

1. *For $\boldsymbol{x} \in \mathbb{R}^\nu$, the function $x_i \in [0,\infty) \mapsto \psi(\boldsymbol{x})$ is non-increasing for any $i \in \{1,\dots,\nu\}$.*
2. *For $\boldsymbol{x} \in \mathbb{R}^\nu$, the function $x_i \in \mathbb{R} \mapsto \psi(\boldsymbol{x})$ is even for any $i \in \{1,\dots,\nu\}$.*

Then

$$2^\nu \sum_{\mathbf{k}\in\mathbb{N}^\nu} \psi(\mathbf{k}) \le \int_{\mathbb{R}^\nu} \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \le 2^\nu \sum_{\mathbf{k}\in\mathbb{N}_0^\nu} \psi(\mathbf{k}).$$

Proof. By condition (1), we see that for any $\mathbf{k} \in \mathbb{N}_0^\nu$, we have

$$\psi(\mathbf{k}+\mathbf{1}) \le \psi(\boldsymbol{x}) \le \psi(\mathbf{k}) \qquad \forall\, \boldsymbol{x} \in [\mathbf{k},\mathbf{k}+\mathbf{1}].$$

Integrating over all $\boldsymbol{x} \in [\mathbf{k},\mathbf{k}+\mathbf{1}]$, summing over all $\mathbf{k} \in \mathbb{N}_0^\nu$ and then multiplying through by 2^ν, we find that

$$2^\nu \sum_{\mathbf{k}\in\mathbb{N}^\nu} \psi(\mathbf{k}) \le 2^\nu \int_{[0,\infty)^\nu} \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \le 2^\nu \sum_{\mathbf{k}\in\mathbb{N}_0^\nu} \psi(\mathbf{k}).$$

Since condition (2) implies that

$$\int_{\mathbb{R}^\nu} \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 2^\nu \int_{[0,\infty)^\nu} \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

the desired result holds. □

Lemma 4.2. *Let $\psi \colon [0,\infty) \to \mathbb{R}$ be a non-increasing function. For any $\nu \in \mathbb{N}$, we have*

$$2^\nu \sum_{\mathbf{k}\in\mathbb{N}^\nu} \psi(\|\mathbf{k}\|_{\ell_2(\mathbb{R}^\nu)}) \le \int_{\mathbb{R}^\nu} \psi(\|\boldsymbol{x}\|_{\ell_2(\mathbb{R}^\nu)})\,\mathrm{d}\boldsymbol{x} \le 2^\nu \sum_{\mathbf{k}\in\mathbb{N}_0^\nu} \psi(\|\mathbf{k}\|_{\ell_2(\mathbb{R}^\nu)}) \tag{4.1}$$

and

$$\int_{\mathbb{R}^\nu} \psi(\|\boldsymbol{x}\|_{\ell_2(\mathbb{R}^\nu)})\,\mathrm{d}\boldsymbol{x} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} \int_0^\infty \rho^{\nu-1}\psi(\rho)\,\mathrm{d}\rho. \tag{4.2}$$

Proof. The proof of (4.1) is similar to the proof of Lemma 4.1. but starting with the inequality

$$\forall\, \mathbf{k} \in \mathbb{N}_0^\nu, \forall\, \boldsymbol{x} \in [\mathbf{k},\mathbf{k}+\mathbf{1}] : \qquad \psi(\|\mathbf{k}+\mathbf{1}\|_{\ell_2(\mathbb{R}^\nu)}) \le \psi(\|\boldsymbol{x}\|_{\ell_2(\mathbb{R}^\nu)}) \le \psi(\|\mathbf{k}\|_{\ell_2(\mathbb{R}^\nu)}).$$

Equation (4.2) is simply formula 4.642 from [13]. □

The main result of this section is about *quasi-polynomial tractability* ("QPT" for short), which was defined in [14]. The problem App_Γ is QPT iff there are positive t and C such that

$$n(\varepsilon, \mathrm{App}_{\Gamma,d}) \le C\, \exp\Big(t(1+\ln\, d)(1+\ln\, \varepsilon^{-1})\Big) \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1].$$

Note that QPT is weaker than the older notions of *polynomial tractability* (PT) or *strong polynomial tractability* (SPT). For PT we require that there be non-negative numbers p, q and C such that

$$n(\varepsilon, \mathrm{App}_{\Gamma,d}) \le C\, \varepsilon^{-p}\, d^{\,q} \qquad \forall\, d \in \mathbb{N}, \varepsilon \in (0,1].$$

By SPT, we mean PT with $q = 0$. Note that SPT $\implies$ PT $\implies$ QPT; the first implication is trivial, and the second holds because

$$\varepsilon^{-p}\, d^{\,q} \le \exp\Big(\max\{p,q\}(1+\ln\, d)(1+\ln\, \varepsilon^{-1})\Big).$$

Hence a problem that isn't QPT is clearly not PT (and certainly not SPT).

Recalling that our weights are positive and bounded by one, we have the following result:

Theorem 4.1. App_Γ *is not quasi-polynomially tractable, no matter how the weights are chosen.*

Proof. From [3, Theorem 23.2], the problem App_Γ is quasi-polynomially tractable iff there exist $\kappa > 0$ and $\tau > 0$ such that

$$\sup_{d\in\mathbb{N}} d^{-\kappa} \sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{d,\mathbf{k}}^{\tau(1+\ln\, d)} < \infty.$$

Hence if we can show that

$$\text{for any } \tau > 0, \text{ we can find } d \in \mathbb{N} \text{ such that } \sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{d,\mathbf{k}}^{\tau(1+\ln\, d)} = \infty, \tag{4.3}$$

then App_Γ is not quasi-polynomially tractable.

So let $\tau > 0$. Define $\gamma_{d,\min} = \min_{|\mathbf{m}|\le r} \gamma_{d,\mathbf{m}}$. Using Lemma 3.2, we see that

$$\begin{aligned}
\lambda_{d,\mathbf{k}}^{-1} &= 1 + \sum_{1\le|\mathbf{m}|\le r} \gamma_{d,\mathbf{m}}^{-1} \prod_{j=1}^{d} k_j^{2m_j} \le 1 + \gamma_{d,\min}^{-1} \sum_{1\le|\mathbf{m}|\le r} \prod_{j=1}^{d} k_j^{2m_j} = 1 + \gamma_{d,\min}^{-1} \sum_{l=1}^{r} \sum_{|\mathbf{m}|=l} \prod_{j=1}^{d} k_j^{2m_j} \\
&\le 1 + \gamma_{d,\min}^{-1} \sum_{l=1}^{r} \Bigl(\sum_{j=1}^{d} k_j^2\Bigr)^{l} \qquad \text{(using the multinomial identity, similarly to [4])} \\
&\le 1 + r\,\gamma_{d,\min}^{-1} \|\mathbf{k}\|_{\ell_2(\mathbb{R}^d)}^{2r}.
\end{aligned}$$

Let $\psi(\rho) = (1 + r\,\gamma_{d,\min}^{-1}\rho^{2r})^{-\tau(1+\ln\, d)}$, which is obviously a decreasing function of $\rho \in [0,\infty)$. Using Lemma 4.2, we have

$$\begin{aligned}
\sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{d,\mathbf{k}}^{\tau(1+\ln\, d)} &\ge \sum_{\mathbf{k}\in\mathbb{N}_0^d} \psi(\|\mathbf{k}\|_{\ell_2(\mathbb{R}^d)}) \ge 2^{-d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \rho^{d-1}\psi(\rho)\,\mathrm{d}\rho \\
&= \frac{\pi^{d/2}}{2^{d-1}\Gamma(d/2)} \int_0^\infty \frac{\rho^{d-1}\,\mathrm{d}\rho}{(1 + r\gamma_{d,\min}^{-1}\rho^{2r})^{\tau(1+\ln\, d)}}.
\end{aligned}$$

For sufficiently large d, we have $d \ge 2r\tau(1 + \ln\, d)$, in which case the integral above is infinite. Thus for any $\tau > 0$, we can find $d \in \mathbb{N}$ such that (4.3) holds, and hence App_Γ is not quasi-polynomially tractable, as claimed. □

5. (s,t)-weak tractability for product weights

We have found that our problem is never QPT, no matter how we choose weights. Can some weaker kind of tractability hold? In particular, we wish to consider conditions under which App_Γ is (s,t)-*weakly tractable* for some positive s and t ("(s,t)-WT" for short), meaning that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\, n(\varepsilon, \mathrm{App}_{\Gamma,d})}{\varepsilon^{-s} + d^{\,t}} = 0,$$

with the convention that $\ln\, 0 = 0$. Hence, (s,t)-WT means that $n(\varepsilon, \mathrm{App}_{\Gamma,d})$ is *not* exponential in ε^{-s} and $d^{\,t}$. This concept was introduced in [5], as a generalization of *weak tractability*, which is simply $(1,1)$-weak tractability. Moreover, a problem that is (s,t)-WT for all positive s and t is said to be *uniformly weakly tractable* (UWT), as defined in [10].

Since $\gamma_{d,\mathbf{m}} \le 1$, our problem is no harder than the analogous approximation problem $\mathrm{App}_d : H^r(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ in the unweighted case, i.e., when all $\gamma_{d,\mathbf{m}} = 1$. From [5], we know that $\mathrm{App} = \{\,\mathrm{App}_d : d \in \mathbb{N}\,\}$ is (s,t)-weakly tractable iff either $rs > 2$ or $t > 1$. Hence, we have the following corollary:

Corollary 5.1. *Let $rs > 2$ or $t > 1$. Then no matter how the (strictly positive) weights are chosen,* App_Γ *is (s,t)-weakly tractable.*

So it will suffice to restrict our attention to the case where $rs \le 2$ and $t \in (0, 1]$. We shall consider *product weights*, which have the form

$$\gamma_{d,\mathbf{m}} = \prod_{j=1}^{d} \gamma_{d,j}^{m_i} \qquad \forall\, \mathbf{m} \in \mathbb{N}_0^d, d \in \mathbb{N}.$$

Note that for product weights, we have $\gamma_{d,\mathbf{0}} = 1$, as required for this paper.

Our goal is to characterize (s, t)-WT in terms of how quickly the *weightlets* $\gamma_{d,j}$ go to zero.

Looking at Lemma 3.2, we see that the eigenvalues of $W_{\Gamma,d}$ now have the form

$$\lambda_{d,\mathbf{k}} = \Big[\sum_{|\mathbf{m}|\le r} \prod_{j=1}^{d} \left(\gamma_{d,j}^{-1} k_j^2\right)^{m_j} \Big]^{-1} \forall\, \mathbf{k} \in \mathbb{Z}^d, d \in \mathbb{N}$$

for such product weights. Observe that $\lambda_{d,\mathbf{k}}$ does not depend on the order of the weightlets $\gamma_{d,j}$. Therefore without loss of generality we may assume that the weightlets are ordered, i.e., for all $d \in \mathbb{N}$ we have

$$0 < \gamma_{d,d} \le \gamma_{d,d-1} \le \cdots \le \gamma_{d,1} \le 1.$$

Based on the multinomial identity and on the criterion for (s, t)-WT presented in [8, Theorem 3.1], we show the following lemma.

Lemma 5.1. *For $c > 0$, $s > 0$ and $t > 0$, let*

$$\mu(c, s, t) = \sup_{d\in\mathbb{N}} \Big[\sigma(c, d, s) \exp\left(-c\, d^t\right)\Big],$$

where

$$\sigma(c, d, s) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \exp\left(-c\widetilde{\lambda}_{d,\mathbf{k}}^{-s/2}\right) \tag{5.1}$$

and

$$\widetilde{\lambda}_{d,\mathbf{k}} = \left(1 + \sum_{j=1}^{d} \gamma_{d,j}^{-1} k_j^2\right)^{-r}.$$

Then App_Γ *is (s, t)-weakly tractable iff*

$$\mu(c, s, t) < \infty \qquad \textit{for all positive } c.$$

Proof. As in [4], we may use the multinomial identity to see that

$$\lambda_{d,\mathbf{k}}^{-1} \le \widetilde{\lambda}_{d,\mathbf{k}}^{-1} \le r!\, \lambda_{d,\mathbf{k}}^{-1}.$$

We complete the proof by using [8, Theorem 3.1]. □

Note that the weightlets are allowed to depend on d. However, our main results are only for *dimension-independent* weightlets, in which $\gamma_{d,j}$ is independent of d, so we shall only consider such weights. Recall our previous assumptions that weightlets are ordered and belong to the interval $(0, 1]$. So in the remainder of this paper, we shall assume that

$$\gamma_{d,j} \equiv \gamma_j \qquad \text{for } j \in \{1, \ldots, d\} \text{ and } d \in \mathbb{N}, \tag{5.2}$$

with

$$0 < \gamma_d \le \gamma_{d-1} \le \cdots \le \gamma_1 \le 1. \tag{5.3}$$

We later distinguish between the cases $rs = 2$ and $rs < 2$. Before doing so, we prove a lemma that will be useful for both of these cases.

Lemma 5.2. *Let* App_Γ *be (s, t)-WT, where $rs \in (0, 2]$ and $t \in (0, 1]$. Then*

$$\gamma_d = o\left(\frac{1}{d^{(2-rs)/(rs)}}\right) \qquad \textit{as } d \to \infty. \tag{5.4}$$

Proof. Since App_Γ is (s,t)-WT, Lemma 5.1 tells us that for all positive c, we have

$$\ln\,\mu(c,s,t) = \sup_{d\in\mathbb{N}}\Big(\ln\,\sigma(c,d,s) - c\,d^t\Big) < \infty, \tag{5.5}$$

where $\sigma(c,d,s)$ is defined by (5.1). Noting that

$$\sum_{j=1}^d \gamma_j^{-1}k_j^2 \le \sum_{j=1}^d \gamma_j^{-1} \qquad \forall\,\mathbf{k}\in\{-1,0,1\}^d.$$

and $\left|\{-1,0,1\}^d\right| = 3^d$, it follows that

$$\sigma(c,d,s) \ge \exp\Bigg[\Bigg(1+\sum_{j=1}^d \gamma_j^{-1}\Bigg)^{rs/2}\Bigg]3^d.$$

Since $t\in(0,1]$, we have

$$\ln\mu(c,s,t) \ge \ln\sigma(c,d,s) - c\,d \ge d\left(\ln 3 - \frac{c\,\Big(1+\sum_{j=1}^d\gamma_j^{-1}\Big)^{rs/2}}{d} - c\right).$$

Since $rs/2 \le 1$, Jensen's inequality and $1\le\gamma_1^{-1}\le\gamma_2^{-1}\le\cdots\le\gamma_d^{-1}$ yield that

$$\Bigg(1+\sum_{j=1}^d\gamma_j^{-1}\Bigg)^{rs/2} \le 1+\Bigg(\sum_{j=1}^d\gamma_j^{-1}\Bigg)^{rs/2} \le 1+d^{\,rs/2}\gamma_d^{-rs/2} \le 2d^{\,rs/2}\gamma_d^{-rs/2}.$$

Therefore

$$\ln\mu(c,s,t) \ge d\left(\ln 3 - c2d^{rs/2-1}\gamma_d^{-rs/2} - c\right) \qquad \forall\,d\in\mathbb{N}.$$

Taking positive $c<\ln 3$, we find that

$$\frac{\ln 3 - c}{2c}\left(1-\frac{\ln\mu(c,s,t)}{d}\right) \le \frac{1}{d^{(2-rs)/2}\,\gamma_d^{rs/2}} \qquad \forall\,d\in\mathbb{N}.$$

Without loss of generality, we may assume that $d\ge\frac12\ln\mu(c,s,t)$. The previous inequality implies that

$$\frac{\ln 3 - c}{4c} \le \frac{1}{d^{(2-rs)/2}\,\gamma_d^{rs/2}}$$

for such d, and so

$$\gamma_d \le \left(\frac{4c}{\ln 3 - c}\right)^{2/(rs)}\,\frac{1}{d^{(2-rs)/(rs)}}.$$

Since c can be arbitrarily small we conclude that $\gamma_d = o(d^{-(2-rs)/(rs)})$, as claimed. □

We now characterize (s,t)-WT for the case $rs=2$.

Theorem 5.1. *Let $rs=2$ and $t\in(0,1]$. Then*

$$\mathrm{App}_\Gamma\ \textit{is } (s,t)\textit{-WT} \qquad \textit{iff} \qquad \gamma_j = \begin{cases} o(1) & \textit{for } t=1,\\ o\Big((\ln j)^{-1}\Big) & \textit{for } t\in(0,1),\end{cases} \qquad \textit{as } j\to\infty.$$

Proof. For necessity, suppose that App_Γ is (s,t)-WT; we shall show that the conditions of Theorem 5.1 hold. Using (5.4) with $rs = 2$, we find that $\gamma_j = o(1)$ as $j \to \infty$. So we only need to look at the case $t \in (0,1)$.

We claim that

$$\limsup_{d\to\infty} \frac{1}{d^t} \sum_{j=1}^d \ln\left(1 + 2\exp(-c\,\gamma_j^{-1})\right) \le c \tag{5.6}$$

for sufficiently small c. Indeed, since $rs = 2$, we have

$$\begin{aligned}
\sigma(c,s,t) &\ge \sum_{\mathbf{k}\in\{-1,0,1\}^d} \exp\left[-c\left(1 + \sum_{j=1}^d \gamma_j^{-1} k_j^2\right)\right] \\
&= \exp(-c) \sum_{\mathbf{k}\in\{-1,0,1\}^d} \prod_{j=1}^d \exp\left(-c\,\gamma_j^{-1} k_j^2\right) \\
&\ge \exp(-c) \prod_{j=1}^d \left(1 + 2\exp(-c\,\gamma_j^{-1})\right).
\end{aligned}$$

Hence for all $d \in \mathbb{N}$, we have

$$\ln\,\mu(c,s,t) \ge \ln\,\sigma(c,d,s) - c\,d^t \ge \sum_{j=1}^d \ln\left(1 + 2\exp(-c\,\gamma_j^{-1})\right) - c - cd^{\,t},$$

which we may rewrite as

$$\frac{1}{d^t} \sum_{j=1}^d \ln\left(1 + 2\exp(-c\,\gamma_j^{-1})\right) \le \frac{\ln\mu(c,s,t) + c}{d^t} + c.$$

Since (5.5) holds, our desired result (5.6) now follows.

Since $c > 0$, we have $2\exp(-c\gamma_j^{-1}) \in (0,2]$. Since $x \in (0,2] \mapsto \ln(1+x)/x$ is a decreasing function, we have $\ln(1+x) \ge c_1 x$ for $x \in (0,2]$, where $c_1 = \frac{1}{2}\ln 3 \doteq 0.549306$. Using this inequality, along with (5.6) and the fact that $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_d$, we see that

$$2c \ge \frac{1}{d^t} \sum_{j=1}^d \ln\left(1 + 2\exp(-c\gamma_j^{-1})\right) \ge 2c_1 \exp(-c\gamma_d^{-1})\, d^{1-t}.$$

for sufficiently large d. Therefore

$$d^{1-t}\exp(-c\gamma_d^{-1}) \le \frac{c}{c_1} \qquad \text{for large } d.$$

Take logarithms to find that

$$(1-t)\ln\,d - c\,\gamma_d^{-1} \le \ln\left(\frac{c}{c_1}\right) \qquad \text{for large } d,$$

and so

$$\gamma_d \le \frac{c}{(1-t)\ln\,d - \ln(c/c_1)} \qquad \text{for large } d. \tag{5.7}$$

Since $t < 1$ and $c > 0$ can be arbitrarily small, we see that $\gamma_d = o\left((\ln d)^{-1}\right)$ as $d \to \infty$, as required.

We now show that the conditions of Theorem 5.1 suffice for App_Γ to be (s,t)-WT. Note that

$$\begin{aligned}
\sigma(c,d,s) &= \sum_{\mathbf{k}\in\mathbb{Z}^d} \exp\left[-c\left(1 + \sum_{j=1}^d \gamma_j^{-1} k_j^2\right)\right] \le \prod_{j=1}^d \sum_{k\in\mathbb{Z}} \exp\left(-c\gamma_j^{-1} k^2\right) \\
&= \prod_{j=1}^d \left[1 + 2\sum_{k=1}^\infty \exp\left(-c\,\gamma_j^{-1} k^2\right)\right] \le \prod_{j=1}^d \left[1 + 2\sum_{k=1}^\infty \exp\left(-c\,\gamma_j^{-1} k\right)\right] \\
&\le \prod_{j=1}^d \left[1 + \frac{2\exp(-c\,\gamma_j^{-1})}{1 - \exp(-c\,\gamma_j^{-1})}\right].
\end{aligned} \tag{5.8}$$

Suppose first that $t = 1$. In this case, $\gamma_j = o(1)$ as $j \to \infty$, and we want to show that App_Γ is $(s, 1)$-WT. Let $\eta > 0$, to be chosen later. There exists $j_\eta \in \mathbb{N}$ such that $\gamma_j \le \eta$ for $j \ge j_\eta$. Thus (5.8) implies that

$$\sigma(c,d,s) \le C_\eta \prod_{j=j_\eta}^{d}\left[1 + \frac{2\exp(-c\,\gamma_j^{-1})}{1-\exp(-c\,\gamma_j^{-1})}\right] \le C_\eta\left[1 + \frac{2\exp(-c\,\eta^{-1})}{1-\exp(-c\,\eta^{-1})}\right]^d,$$

where

$$C_\eta = \prod_{j=1}^{j_\eta - 1}\left[1 + \frac{2\exp(-c\,\gamma_j^{-1})}{1-\exp(-c\,\gamma_j^{-1})}\right].$$

Since $\ln(1+x) \le x$ for positive x, we have

$$\ln\,\sigma(c,d,s) \le \ln\,C_\eta + d\ln\left(1 + \frac{2\exp(-c\,\eta^{-1})}{1-\exp(-c\,\eta^{-1})}\right) \le \ln\,C_\eta + d\left(\frac{2\exp(-c\,\eta^{-1})}{1-\exp(-c\,\eta^{-1})}\right) \tag{5.9}$$

Choose

$$\eta = \frac{c}{\ln\dfrac{2+c}{c}},$$

so that

$$\frac{2\exp(-c\,\eta^{-1})}{1-\exp(-c\,\eta^{-1})} = c. \tag{5.10}$$

Using (5.9)–(5.10), we see that

$$\ln\,\mu(c,s,1) = \sup_{d\in\mathbb{N}}(\ln\,\sigma(c,d,s) - c\,d) \le \sup_{d\in\mathbb{N}}\Big((\ln\,C_\eta + c\,d) - c\,d\Big) = \ln\,C_\eta < \infty,$$

and hence App_Γ is $(s, 1)$-WT, as claimed.

Now suppose that $t < 1$. In this case, $\gamma_j = o((\ln\,j)^{-1})$ as $j \to \infty$. We want to show that App_Γ is (s, t)-WT. Let $\eta > 0$, to be chosen later. There exists $j_\eta \in \mathbb{N}$ with $j_\eta \ge 2$, such that $\gamma_j \le \eta/\ln\,j$ for $j \ge j_\eta$. Then

$$\frac{2\exp(-c\,\gamma_j^{-1})}{1-\exp(-c\,\gamma_j^{-1})} \le \frac{2\exp(-c\,\eta^{-1}\ln\,j)}{1-\exp(-c\,\eta^{-1}\ln\,j)} = \frac{2j^{-c/\eta}}{1-j^{-c/\eta}} \qquad \text{for } j \ge j_\eta.$$

Letting

$$C_\eta = \prod_{j=1}^{j_\eta-1}\left(1 + \frac{2j^{-c/\eta}}{1-j^{-c/\eta}}\right),$$

we see from (5.8) that

$$\sigma(c,d,s) \le C_\eta \prod_{j=j_\eta}^{d}\left(1 + \frac{2j^{-c/\eta}}{1-j^{-c/\eta}}\right).$$

Taking logarithms, we find that

$$\ln\,\sigma(c,d,s) \le \ln\,C_\eta + 2\sum_{j=j_\eta}^{d}\frac{j^{-c/\eta}}{1-j^{-c/\eta}}.$$

In particular, if we choose $\eta = \frac{1}{2}c$, then $1 - j^{-c/\eta} \ge \frac{3}{4}$, and so we have

$$\ln\,\sigma(c,d,s) \le \ln\,C_\eta + \tfrac{8}{3}\sum_{j=j_\eta}^{d} j^{-2} \le \ln\,C_\eta + \tfrac{8}{3}\sum_{j=1}^{\infty} j^{-2} = \ln\,C_\eta + \tfrac{4}{9}\pi^2.$$

Hence

$$\ln\mu(c,s,t) = \sup_{d\in\mathbb{N}}\Big(\ln\,\sigma(c,d,s) - c\,d^t\Big) \le \sup_{d\in\mathbb{N}}\ln\,\sigma(c,d,s) \le \ln\,C_\eta + \tfrac{4}{9}\pi^2 < \infty,$$

and so App_Γ is (s, t)-WT, as required. $\square$

We now take up the case $rs < 2$.

Theorem 5.2. *Let $rs < 2$ and $t \in (0,1]$. Then*

$$\mathrm{App}_\Gamma \text{ is } (s,t)\text{-WT} \qquad \text{iff} \qquad \gamma_j = o(j^{-(2-rs)/(rs)}) \text{ as } j \to \infty.$$

Proof. In Lemma 5.2, we showed that the condition on $\{\gamma_j\}_{j\in\mathbb{N}}$ was necessary. Therefore we only need to prove the sufficiency of this condition.

Let $\alpha := rs/2 < 1$. For $c > 0$ and $d \in \mathbb{N}$, let

$$\widetilde{\sigma}(c,d) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \exp\Big[-c\Big(1 + \sum_{j=1}^{d} \gamma_j^{-1} k_j^2\Big)^{\alpha}\Big].$$

By Lemma 5.1, the problem App_Γ is (s,t)-WT iff

$$\widetilde{\mu}(c,t) < \infty \qquad \text{for all positive } c, \tag{5.11}$$

where

$$\widetilde{\mu}(c,t) = \sup_{d\in\mathbb{N}} \Big[\widetilde{\sigma}(c,d)\, \exp\big(-c\, d^t\big)\Big].$$

Since $\gamma_j = o(j^{-(2-rs)/(rs)})$, it follows that for any $\kappa > 0$ and any $M \ge \kappa/c$ there exists $\ell = \ell(M) \in \mathbb{N}$ such that

$$\frac{1}{\gamma_j} \ge \begin{cases} 1 & \text{for } j \le \ell, \\ M^{1/\alpha} j^{(1-\alpha)/\alpha} & \text{for } j > \ell. \end{cases}$$

We will determine the value of κ later on.

We need to get an upper bound on $\widetilde{\sigma}(c,d)$. Since $\widetilde{\sigma}(c,d)$ is an increasing function of d, it is enough to consider $d > \ell$. Using the inequality

$$(\xi_1 + \xi_2)^\alpha \ge \frac{1}{2^{1-\alpha}}\left(\xi_1^\alpha + \xi_2^\alpha\right) \qquad \forall\, \xi_1, \xi_2 \ge 0, \tag{5.12}$$

we find that

$$\widetilde{\sigma}(c,d) \le \sum_{\mathbf{k}\in\mathbb{Z}^d} \exp\Big[\frac{-c}{2^{1-\alpha}}\Big(1 + \sum_{j=1}^{\ell} k_j^2\Big)^{\alpha} + \frac{-cM}{2^{1-\alpha}}\Big(\sum_{j=\ell+1}^{d} k_j^2 j^{(1-\alpha)/\alpha}\Big)^{\alpha}\Big]$$

Letting

$$C_\ell = \sum_{\mathbf{k}\in\mathbb{Z}^\ell} \exp\left[\frac{-c}{2^{1-\alpha}}\left(1 + \sum_{j=1}^{\ell} k_j^2\right)^{\alpha}\right],$$

which is finite and independent of d, we may rewrite the previous inequality as

$$\widetilde{\sigma}(c,d) \le C_\ell \sum_{\mathbf{k}\in\mathbb{Z}^{d-\ell}} \exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j=1}^{d-\ell} k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^{\alpha}\Big].$$

Define

$$V_{d-\ell,n} = \{\, \mathbf{k} \in \mathbb{Z}^{d-\ell} : \mathbf{k} \text{ has } n \text{ nonzero components} \,\} \qquad \forall\, n \in \{0,1,\dots,d-\ell\}.$$

For $\mathfrak{u} \subseteq \{1,2,\dots,d-\ell\}$, let $\mathbb{N}^{\mathfrak{u}}$ denote n-dimensional integer vectors, indexed by the elements of $\mathfrak{u}$, so that for $\mathfrak{u} = \{j_1, j_2, \dots, j_{d-\ell}\}$, we have

$$\mathbf{k} = [k_j]_{j\in\mathfrak{u}} = [k_{j_1}, k_{j_2}, \dots, k_{j_{d-\ell}}] \qquad \forall\, \mathbf{k} \in \mathbb{N}^{\mathfrak{u}}.$$

We then have

$$\begin{aligned}
\widetilde{\sigma}(c,d) &\le C_\ell \sum_{n=0}^{d-\ell} \sum_{\mathbf{k}\in V_{d-\ell,n}} \exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j=1}^{d-\ell} k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^{\alpha}\Big] \\
&\le C_\ell \sum_{n=0}^{d-\ell} \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\dots,d-\ell\} \\ |\mathfrak{u}|=n}} 2^n \sum_{\mathbf{k}\in\mathbb{N}^{\mathfrak{u}}} \exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j\in\mathfrak{u}} k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^{\alpha}\Big].
\end{aligned}$$

Hence, we obtain

$$\widetilde{\sigma}(c,d) \le C_\ell \sum_{n=0}^{d-\ell} a_n 2^n \binom{d-\ell}{n}, \tag{5.13}$$

where

$$a_n = \begin{cases} \displaystyle\sum_{\mathbf{k}\in\mathbb{N}^n} \exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j=1}^n k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big] & \text{for } n \in \{1,\dots,d-\ell\},\\ 1 & \text{for } n = 0, \end{cases}$$

because for any fixed $\mathfrak{u}$ with $|\mathfrak{u}| = n > 1$, we have

$$\exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j=1}^n k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big] \le a_n.$$

Estimating the binomial coefficient in (5.13) by $d^n/n!$, we have

$$\widetilde{\sigma}(c,d) \le C_\ell \sum_{n=0}^{d-\ell} a_n 2^n \frac{d^n}{n!}. \tag{5.14}$$

We now estimate a_n for $n \in \{1,2,\dots,d-\ell\}$. Letting

$$W_{n,m} = \{\, \mathbf{k} \in \mathbb{N}^n : n-m \text{ of } \mathbf{k}\text{'s components equal } 1 \,\} \qquad \text{for } m \in \{0,1,\dots,n\},$$

and noting that $k^2 \ge (k-1)^2 + 1$ for $k \in \mathbb{N}$, we have

$$\begin{aligned}
a_n &= \sum_{m=0}^n \sum_{\mathbf{k}\in W_{n,m}} \exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j=1}^n k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big]\\
&= \sum_{m=0}^n \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\dots,n\}\\ |\mathfrak{u}|=m}} \sum_{\mathbf{k}\in(\mathbb{N}\setminus\{1\})^{\mathfrak{u}}} \exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j\notin\mathfrak{u}} (j+\ell)^{(1-\alpha)/\alpha} + \sum_{j\in\mathfrak{u}} k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big]\\
&\le \sum_{m=0}^n \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\dots,n\}\\ |\mathfrak{u}|=m}} \sum_{\mathbf{k}\in(\mathbb{N}\setminus\{1\})^{\mathfrak{u}}} \exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j\notin\mathfrak{u}} (j+\ell)^{(1-\alpha)/\alpha} + \sum_{j\in\mathfrak{u}} [(k_j-1)^2+1] (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big]\\
&= \sum_{m=0}^n \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\dots,n\}\\ |\mathfrak{u}|=m}} \sum_{\mathbf{k}\in\mathbb{N}^{\mathfrak{u}}} \exp\Big[\frac{-cM}{2^{1-\alpha}}\Big(\sum_{j=1}^n (j+\ell)^{(1-\alpha)/\alpha} + \sum_{j\in\mathfrak{u}} k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big]\\
&\le \sum_{m=0}^n \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\dots,n\}\\ |\mathfrak{u}|=m}} \sum_{\mathbf{k}\in\mathbb{N}^{\mathfrak{u}}} \exp\left[\frac{-cM}{2^{2(1-\alpha)}}\Big[\Big(\sum_{j=1}^n (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha + \Big(\sum_{j\in\mathfrak{u}} k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big]\right],
\end{aligned}$$

where we have used (5.12) in the last step above. Letting

$$b_m = \sum_{\mathbf{k}\in\mathbb{N}^m} \exp\Big[\frac{-cM}{4^{1-\alpha}}\Big(\sum_{j=1}^m k_j^2 (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big]$$

and

$$c_n = \exp\Big[\frac{-cM}{4^{1-\alpha}}\Big(\sum_{j=1}^n (j+\ell)^{(1-\alpha)/\alpha}\Big)^\alpha\Big] \qquad \text{for } n \in \{1,\dots,d-\ell\},$$

we have

$$a_n \le c_n \sum_{m=0}^n b_m \binom{n}{m} \le 2^n c_n \sup_{m\in\mathbb{N}_0} b_m. \tag{5.15}$$

First, let us estimate c_n. We have

$$\Big(\sum_{j=1}^{n}(j+\ell)^{(1-\alpha)/\alpha}\Big)^{\alpha} \geq \Big(\int_0^n (x+\ell)^{(1-\alpha)/\alpha}\,\mathrm{d}x\Big)^{\alpha} = \alpha^{\alpha}(n+\ell)\bigg(1-\Big(\frac{\ell}{n+\ell}\Big)^{1/\alpha}\bigg)^{\alpha}.$$

Since $n \geq 1$, there exists $C_\alpha > 0$, independent of n, such that

$$\Big(\sum_{j=1}^{n}(j+\ell)^{(1-\alpha)/\alpha}\Big)^{\alpha} \geq C_\alpha(n+\ell),$$

and so

$$c_n \leq \exp(-\omega n) \quad \text{with} \quad \omega = \frac{cM}{4^{1-\alpha}}\,C_\alpha. \tag{5.16}$$

We next show that $b := \sup_{m\in\mathbb{N}_0} b_m$ is finite. Indeed, we may use Lemma 4.1 to see that

$$\begin{aligned} b_m &\leq 2^{-m}\int_{\mathbb{R}^m}\exp\Big[\frac{-cM}{4^{1-\alpha}}\Big(\sum_{j=1}^{m}(j+\ell)^{(1-\alpha)/\alpha}x_j^2\Big)^{\alpha}\Big]\,\mathrm{d}\boldsymbol{x} \\ &= 2^{-m}\Big[\frac{4^{1-\alpha}}{cM}\Big]^{m/(2\alpha)}\Big[\frac{\ell!}{(\ell+m)!}\Big]^{(1-\alpha)/(2\alpha)}\int_{\mathbb{R}^m}\exp\Big[-\Big(\sum_{j=1}^{m}t_j^2\Big)^{\alpha}\Big]\,\mathrm{d}\boldsymbol{t}, \end{aligned}$$

where we have used the change of variables

$$t_j = \Big(\frac{cM}{4^{1-\alpha}}\Big)^{1/(2\alpha)}(j+\ell)^{(1-\alpha)/(2\alpha)}x_j \qquad (1 \leq j \leq m).$$

Using Lemma 4.2, we have

$$\int_{\mathbb{R}^m}\exp\Big[-\Big(\sum_{j=1}^{m}t_j^2\Big)^{\alpha}\Big]\,\mathrm{d}\mathbf{t} = \frac{2\pi^{m/2}}{\Gamma(m/2)}\int_0^\infty x^{m-1}\exp(-x^{2\alpha})\,\mathrm{d}x$$

Using the change of variables $t = x^{2\alpha}$, we have

$$\int_0^\infty x^{m-1}\exp(-x^{2\alpha})\,\mathrm{d}x = \frac{1}{2\alpha}\int_0^\infty t^{(m-1)/(2\alpha)+1/(2\alpha)-1}\exp(-t)\,\mathrm{d}t = \frac{1}{2\alpha}\Gamma\Big(\frac{m}{2\alpha}\Big),$$

so that

$$\int_{\mathbb{R}^m}\exp\Big[-\Big(\sum_{j=1}^{m}t_j^2\Big)^{\alpha}\Big]\,\mathrm{d}\mathbf{t} = \frac{\pi^{m/2}}{\alpha}\,\frac{\Gamma\Big(\dfrac{m}{2\alpha}\Big)}{\Gamma\Big(\dfrac{m}{2}\Big)},$$

and hence

$$b_m \leq \frac{1}{\alpha}\Big(\frac{4^{1-2\alpha}\pi^{\alpha}}{cM}\Big)^{m/(2\alpha)}\Big(\frac{\ell!}{(m+\ell)!}\Big)^{(1-\alpha)/(2\alpha)}\,\frac{\Gamma\Big(\dfrac{m}{2\alpha}\Big)}{\Gamma\Big(\dfrac{m}{2}\Big)}. \tag{5.17}$$

It is well-known that

$$\ln\frac{\Gamma\Big(\dfrac{m}{2\alpha}\Big)}{\Gamma\Big(\dfrac{m}{2}\Big)} = \Big(\frac{1}{2\alpha}-\frac{1}{2}\Big)(m\ln m) + \mathscr{O}(m) \qquad \text{as } m\to\infty,$$

where the $\mathscr{O}$-factor is independent of m. Furthermore,

$$\ln\Big(\frac{\ell!}{(m+\ell)!}\Big)^{(1-\alpha)/(2\alpha)} = \Big(\frac{1}{2\alpha}-\frac{1}{2}\Big)(-m\ln m) + \mathscr{O}(m) \qquad \text{as } m\to\infty,$$

where the $\mathscr{O}$-factor is once again independent of m. Therefore

$$\ln b_m = -\frac{m}{2\alpha}\ \ln(cM) + \mathscr{O}(m) \qquad \text{as } m \to \infty.$$

Since $M \geq \kappa/c$, the last right-hand side can be upper bounded by $-m/(2\alpha)\ln(\kappa) + Cm$ for some positive C, Hence if we choose $\kappa > \exp(2C\alpha)$, then $\ln b_m$ goes to $-\infty$ with m. Hence, $b = \sup_m b_m < \infty$, as claimed.

From (5.15) and (5.16), we find that

$$a_n \leq b \cdot 2^n \exp(-\omega\, n) \qquad \text{with} \qquad \omega = \frac{cM}{4^{1-\alpha}} C_\alpha. \tag{5.18}$$

We now substitute (5.18) back into (5.14), getting

$$\widetilde{\sigma}(c,d) \leq b\, C_\ell \sum_{n=0}^{d} \beta_n, \tag{5.19}$$

where

$$\beta_n = \frac{(4d)^n}{n!} \mathrm{e}^{-\omega n}.$$

Note that

$$\frac{\beta_{n+1}}{\beta_n} = \frac{4d}{(n+1)\mathrm{e}^{\omega}},$$

which is a decreasing function of $n \in \mathbb{N}_0$. Hence the maximum value of β_n would occur when $\beta_{n+1}/\beta_n = 1$, i.e., when $n = \tilde{n} := 4d\mathrm{e}^{-\omega} - 1$. Since $\tilde{n}$ might not be an integer, the maximum value of β_n will occur at β_{n^*}, where n^* is either $\lfloor \tilde{n} \rfloor$ or $\lceil \tilde{n} \rceil$. In any case, we have

$$n^* = 4d\mathrm{e}^{-\omega}\left(1 + \mathscr{O}(d^{-1})\right) \qquad \text{as } d \to \infty,$$

and

$$\widetilde{\sigma}(c,d) \leq b\, C_\ell\, (d+1)\beta_{n^*} = b\, C_\ell\, (d+1)\, \frac{(4d)^{n^*}}{(n^*)!}\, \mathrm{e}^{-\omega\, n^*}.$$

Hence,

$$\begin{aligned}
\ln \widetilde{\sigma}(c,d) &= \ln d + n^* \ln(4d) - \ln(n^*!) - \omega\, n^* + \mathscr{O}(1) \\
&= \left[4d\mathrm{e}^{-\omega}\ \ln(4d) - 4d\mathrm{e}^{-\omega}(\ln(4d\mathrm{e}^{-\omega}) + \omega)\right] + \mathscr{O}(\ln d) \\
&= \mathscr{O}(\ln d),
\end{aligned}$$

the last line holding because the terms in the square brackets cancel each other out.

Since $t > 0$, this proves that $\widetilde{\sigma}(c,d)\exp(-c\, d^t)$ is uniformly bounded in d. Thus condition (5.11) is satisfied and so we see that App_Γ is (s,t)-WT, as required. □

6. Uniform weak tractability

Based on the results of the previous section it is easy to conclude what are necessary and sufficient conditions on product weights to obtain UWT (uniform weak tractability), see [10]. Recall, that App_Γ is UWT iff it is (s,t)-WT for all positive s and t.

Theorem 6.1. *Consider* App_Γ *with product weights and with* $0 < \gamma_d \leq \gamma_{d-1} \leq \cdots \gamma_1 \leq 1$. *Then*

$$\mathrm{App}_\Gamma \textit{ is UWT} \qquad \textit{iff} \qquad \gamma_j = o(j^{-p}) \textit{ as } j \to \infty \textit{ for all } p > 0.$$

Proof. Obviously, it is enough to consider only large p. In what follows we assume that $p > 1$.

Suppose that App_Γ is UWT. Then App_Γ is (s,t)-WT for $s = 2/(r(p+1))$ and $t \in (0,1]$. Note that $rs = 2/(p+1) < 1$. Since $(2-rs)/(rs) = p$, Theorem 5.2 yields that $\gamma_j = o(j^{-p})$ as $j \to \infty$. By varying s, we can make p arbitrarily large, as needed.

Now suppose that $\gamma_j = o(j^{-p})$ as $j \to \infty$ for all $p > 1$. Then for $s = 2/(r(p+1))$ and $t \in (0,1]$ we have $\gamma_j = o(j^{-(2-rs)/(rs)})$ as $j \to \infty$. Thus Theorem 5.2 yields that App_Γ is (s,t)-WT. By varying p we can obtain arbitrarily small s, and thus UWT holds. □

Hence, UWT holds iff γ_j goes faster to zero than any power of j. For instance, if $\gamma_j = \exp(-j)$, we have UWT.

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