Tractability of Approximation For Some Weighted Spaces of Hybrid Smoothness

Arthur G. Werschulz

Dedicated to our friend and colleague Ian H. Sloan on the occasion of his 80th birthday.

Abstract A great deal of work has studied the tractability of approximating (in the L_2-norm) functions belonging to weighted unanchored Sobolev spaces of dominating mixed smoothness of order 1 over the unit d-cube. In this paper, we generalize these results. Let r and s be non-negative integers, with $r \le s$. We consider the approximation of complex-valued functions over the torus $\mathbb{T}^d = [0, 2\pi]^d$ from weighted spaces $H_\Gamma^{s,1}(\mathbb{T}^d)$ of hybrid smoothness, measuring error in the $H^r(\mathbb{T}^d)$-norm. Here we have isotropic smoothness of order s, the derivatives of order s having dominating mixed smoothness of order 1. If $r = s = 0$, then $H^{0,1}(\mathbb{T}^d)$ is a well-known weighted unachored Sobolev space of dominating smoothness of order 1, whereas we have a generalization for other values of r and s. Besides its independent interest, this problem arises (with $r = 1$) in Galerkin methods for solving second-order elliptic problems. Suppose that continuous linear information is admissible. We show that this new approximation problem is topologically equivalent to the problem of approximating $H_\Gamma^{s-r,1}(\mathbb{T}^d)$ in the $L_2(\mathbb{T}^d)$-norm, the equivalence being independent of d. It then follows that our new problem attains a given level of tractability if and only if approximating $H_\Gamma^{s-r,1}(\mathbb{T}^d)$ in the $L_2(\mathbb{T}^d)$-norm has the same level of tractability. We further compare the tractability of our problem to that of $L_2(\mathbb{T}^d)$-approximation for $H_\Gamma^{0,1}(\mathbb{T}^d)$. We then analyze the tractability of our problem for various families of weights.

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1 Introduction

Much recent research in information-based complexity has dealt with the issue of tractability. To what extent is it computationally feasible to solve this problem? To get an idea of the scope of this area, see the three-monograph series [7, 8, 9]. Most of the work in this area has dealt with the integration problem (which was the initial impetus for studying tractability in the first place) and the approximation problem (mainly in L_p-norms, with most of the work for the case $p = 2$). This paper deals with the latter.[1]

It has long been known that the L_2-approximation problem for the unit ball of $H^s(I^d)$ over the unit d-cube I^d has nth minimal error $\Theta(n^{-s/d})$, so that $\Theta(\varepsilon^{-d/s})$ information evaluations are necessary and sufficient for an ε-approximation. Were it not for the Θ-factors, this would imply that this problem suffers from what Richard Bellman [1] called "the curse of dimensionality", i.e., an exponential dependence on the dimension d. It turns out that things are not quite as bad as this. To avoid some technical difficulties, we'll use spaces $H^s(\mathbb{T}^d)$ defined over the d-torus $\mathbb{T}^d$, rather than over the d-dimensional unit cube I^d. Kühn *et al.* [5] showed that the Θ-factors decay polynomially in d, and that this problem does not suffer from the curse of dimensionality.

However, we would much prefer something stronger; in particular, we would like to have polynomial tractability, with nth minimal error at most $Cd^q\varepsilon^{-p}$ for C p, and q independent of ε and d or (better yet) strong polynomial tractability, with nth minimal error at most $C\varepsilon^{-p}$ for C and p independent of ε and d. However, the results in [5] imply that the aforementioned problem is not polynomially tractable. So if we want a better tractability result, we need to change the space of functions being approximated.

Now the spaces $H^s(\mathbb{T}^d)$ are isotropic—all variables are equally important. This has led many authors to use anisotropic spaces. In particular, we have used weighted spaces that (algebraically) are tensor products of $H^1(I)$, with the weight family Γ entering into the norm. These are weighted versions of spaces having *mixed smoothness*, as per [6]. In [11], we were able to find conditions on certain weights families Γ that were necessary and sufficient for the $L_2(\mathbb{T}^d)$-approximation problem to be (strongly) polynomially tractable.

We would like to extend these results to weighted spaces of *hybrid* smoothness, see [10]. These are weighted versions of the spaces $H^{s_1,s_2}(\mathbb{T}^d)$, the members of which being periodic functions having isotropic smoothness of order s_1 and dominating mixed smoothness of order s_2.

In this paper, we make a first step in such a study. We will consider spaces $H^{s,1}_\Gamma(\mathbb{T}^d)$. Functions belonging to this space have Sobolev derivatives of order s, said derivatives themselves having one derivative in each coordinate direction. The weights only apply to the anisotropic part of the $H^{s,1}(\mathbb{T}^d)$-norm. We measure error in the $H^r(\mathbb{T}^d)$-sense. Here r and s are non-negative integers, with $r \le s$.

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[1] This introduction is merely an overview. Precise definitions are given in Section 2.

We have an ulterior motive for studying these particular spaces. Suppose we are trying to solve the elliptic problem $-\Delta u + qu = f$ over $\mathbb{T}^d$, with f, q in the unit ball of $H^{0,1}(\mathbb{T}^d)$. Suppose further that we have an elliptic regularity result, saying that $u \in H^{2,1}_\Gamma(\mathbb{T}^d)$ for $f, q \in H^{0,1}_\Gamma(\mathbb{T}^d)$. Then the error of a Galerkin method using an optimal test/trial space will roughly be the minimal error for the $H^1(\mathbb{T}^d)$-approximation problem over $H^{2,1}(\mathbb{T}^d)$. This explains our interest in the $H^r(\mathbb{T}^d)$-approximation problem for $H^{s,1}_\Gamma(\mathbb{T}^d)$ with $r = 1$ and $s = 2$. In this paper, we study the general case (with $r \le s$), which is as easy to handle as the special case $r = 1$ and $s = 2$. In addition, we expect the results of this paper to hold for negative r; this is important because the case $r = -1$ occurs in non-regular second-order elliptic problems, see (e.g.) [3] for further discussion.

The overall structure of this paper is as follows. In Section 2, we precisely define the terminology surrounding the problem we're trying to solve. The results we seek depend on spectral information of a particular linear operator on $H^{s,1}_\Gamma(\mathbb{T}^d)$, which we give in Section 3. Finally, Section 4 gives the tractability results for our approximation problem:

1. If $\mathrm{App}_{\Gamma,0,0}$ has a given level of tractability, then $\mathrm{App}_{\Gamma,r,s}$ has at least the same level of tractability, and the exponent(s) for $\mathrm{App}_{\Gamma,r,s}$ are bounded from above by those for $\mathrm{App}_{\Gamma,0,0}$.
2. Under certain boundedness conditions, $\mathrm{App}_{\Gamma,r,s}$ has a given level of tractability iff $\mathrm{App}_{\Gamma,0,0}$ has at least the same level of tractability. We give estimates relating the exponents for these two problems.
3. For the unweighted case, $\mathrm{App}_{\Gamma,r,s}$ is quasi-polynomially tractable, with exponent $2/\ln 2 \doteq 2.88539$.
4. For bounded product weights:
   a. $\mathrm{App}_{\Gamma,r,s}$ is always quasi-polynomially tractable. We give an estimate of the exponent.
   b. We give conditions on the weights that are necessary and sufficient to guarantee (strong) polynomial tractability, along with estimates of the exponents.
5. For bounded finite-order and finite-diameter weights, $\mathrm{App}_{\Gamma,r,s}$ is always polynomially tractable. We give estimates for the exponents.

2 Problem Definition

In this section, we define the approximation problem to be studied and recall some basic concepts of information-based complexity.

First, we establish some notational conventions. We let $\mathbb{N}$ denote the strictly positive integers, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denoting the natural numbers. (As usual, we let $\mathbb{Z}$ denote the integers.) Next, we let $\mathbb{T}$ denote the torus $[0, 2\pi]$, so that $\mathbb{T}^d$ is the d-torus. We identify opposite points on the d-torus, so that for any $f\colon \mathbb{T}^d \to \mathbb{C}$, we have $f(\boldsymbol{x}) = f(\boldsymbol{y})$ whenever $\boldsymbol{x} - \boldsymbol{y} \in 2\pi\mathbb{Z}^d$. In this sense, functions on the d-torus are periodic. We denote points in $\mathbb{R}^d$ by boldface italic letters, and points in $\mathbb{Z}^d$ (including multi-indices) by boldface roman letters. The unit ball of a normed space X is denoted by $\mathscr{B}X$. Any product over the empty set is defined to be the appropriate multiplicative identity.

We now describe some Sobolev spaces, see (e.g.) [4, 5, 10, 13] for further discussion. Let $L_2(\mathbb{T}^d)$ denote the space of complex-valued square-integrable functions over $\mathbb{T}^d$ and let $r \in \mathbb{N}_0$. Then

$$H^r(\mathbb{T}^d) = \left\{ f \in L_2(\mathbb{T}^d) : \; D^{\mathbf{m}} f \in L_2(\mathbb{T}^d) \;\text{ for }\; |\mathbf{m}| \le r \right\},$$

is the (classical) isotropic Sobolev space of order r, which is a Hilbert space under the usual inner product

$$\langle f, g \rangle_{H^r(\mathbb{T}^d)} = \sum_{|\mathbf{m}| \le r} \langle D^{\mathbf{m}} f, D^{\mathbf{m}} g \rangle_{L_2(\mathbb{T}^d)}.$$

Here, for $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_0^d$, we write

$$D^{\mathbf{m}} = \prod_{j=1}^{d} \frac{\partial^{m_j}}{\partial x_j^{m_j}} \qquad \text{and} \qquad \boldsymbol{z}^{\mathbf{m}} = \prod_{j=1}^{d} z_j^{m_j} \quad \forall \boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d,$$

as well as $|\mathbf{m}| = \sum_{j=1}^{d} m_j$. Here, the partial derivative $\partial/\partial x_j$ is in the distributional sense.

For $s \in \mathbb{N}_0$, we define the space[2]

$$H^{s,1}(\mathbb{T}^d) = \{ v \in H^s(\mathbb{T}^d) : \partial_{\mathfrak{u}} v \in H^s(\mathbb{T}^d) \text{ for all } \mathfrak{u} \subseteq [d] \}$$

of hybrid smoothness, which is a Hilbert space under the inner product

$$\langle v, w \rangle_{H^{s,1}(\mathbb{T}^d)} = \sum_{\mathfrak{u} \subseteq [d]} \langle \partial_{\mathfrak{u}} v, \partial_{\mathfrak{u}} w \rangle_{H^s(\mathbb{T}^d)} \qquad \forall v, w \in H^{s,1}_{\Gamma}(\mathbb{T}^d).$$

Here, we write

$$\partial_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \frac{\partial}{\partial x_i} \qquad \forall \mathfrak{u} \subseteq [d],$$

where $[d] := \{1, 2, \ldots, d\}$.

Our final Sobolev space is a weighted version of the space $H^{s,1}(\mathbb{T}^d)$. Let

$$\Gamma = \{ \gamma_{d,\mathfrak{u}} \ge 0 : \mathfrak{u} \subseteq [d], d \in \mathbb{N} \}$$

be a given set of non-negative weights $\gamma_{d,\mathfrak{u}}$, with $\gamma_{\mathfrak{u},\emptyset} = 1$ for all $d \in \mathbb{N}$. Then we let $H^{s,1}_{\Gamma}(\mathbb{T}^d)$ be $H^{s,1}(\mathbb{T}^d)$, but under the inner product

[2] The superscript 1 in $H^{s,1}(\mathbb{T}^d)$ means that we are taking dominating mixed derivatives of order 1.

$$\langle v,w\rangle_{H^{s,1}_{\Gamma}(\mathbb{T}^d)} = \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}>0}} \gamma_{d,\mathfrak{u}}^{-1}\langle \partial_{\mathfrak{u}} v, \partial_{\mathfrak{u}} w\rangle_{H^s(\mathbb{T}^d)} \qquad \forall\, v,w \in H^{s,1}_{\Gamma}(\mathbb{T}^d). \tag{1}$$

Clearly $H^{s,1}_{\Gamma}(\mathbb{T}^d)$ is a Hilbert space under this inner product.

We now describe the problem we wish to solve. Let $r,s\in\mathbb{N}_0$, with $r\le s$. Our goal is to approximate functions from $\mathscr{B}H^{s,1}_{\Gamma}(\mathbb{T}^d)$ in the $H^r(\mathbb{T}^d)$-norm. This approximation problem is described by the embedding operator $\mathrm{App}_{d,\Gamma,r,s}\colon H^{s,1}_{\Gamma}(\mathbb{T}^d)\to H^r(\mathbb{T}^d)$, which is defined as

$$\mathrm{App}_{d,\Gamma,r,s} f = f \qquad \forall f\in H^{s,1}_{\Gamma}(\mathbb{T}^d).$$

Remark 1. We note some special cases of this problem:

1. Suppose that $r=s=0$. Then $\mathrm{App}_{d,\Gamma,r,s}=\mathrm{App}_{d,\Gamma,0,0}$, and our problem is that of approximating functions from $\mathscr{B}H^{0,1}_{\Gamma}(\mathbb{T}^d)$ in the $L_2(\mathbb{T}^d)$-norm. This problem is analogous to the problem that was extensively covered in [11], the main difference being that [11] dealt with functions defined over the unit cube, rather than the unit torus.
2. Let $\Gamma(\emptyset)$ be given by

$$\gamma_{d,\mathfrak{u}} = \begin{cases} 1 & \text{if } \mathfrak{u}=\emptyset,\\ 0 & \text{otherwise.}\end{cases} \qquad \forall\,\mathfrak{u}\subseteq[d], d\in\mathbb{N}.$$

   Allowing a slight abuse of language, we call $\Gamma(\emptyset)$ *empty weights*. Then $\mathrm{App}_{d,\Gamma,r,s} = \mathrm{App}_{d,\Gamma(\emptyset),r,s}$, and our problem is that of approximating functions from $\mathscr{B}H^s(\mathbb{T}^d)$ in the $H^r(\mathbb{T}^d)$-norm. This problem was studied for the case $r=0$ in [5, 13] and for arbitrary $r\ge 0$ in [10].
3. Let $\Gamma(\text{UNW})$ be defined as

$$\gamma_{d,\mathfrak{u}} = 1 \qquad \forall\,\mathfrak{u}\subseteq[d].$$

   Then $\mathrm{App}_{d,\Gamma,r,s}=\mathrm{App}_{d,\Gamma(\text{UNW}),r,s}$ and we are trying to solve the *unweighted* case. Our problem is now that of approximating functions from $\mathscr{B}H^{s,1}(\mathbb{T}^d)$ in the $H^r(\mathbb{T}^d)$-norm. A non-periodic version of this problem (over the unit cube, rather than the torus) was discussed in [11, Section 4.1.1].
4. If $\Gamma(\Pi)$ is a set of weights defined by

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \qquad \forall\,\mathfrak{u}\subseteq[d], d\in\mathbb{N}, \tag{2}$$

   where

$$\gamma_{d,1}\ge\gamma_{d,2}\ge\cdots\ge\gamma_{d,d}>0 \qquad \forall d\in\mathbb{N}, \tag{3}$$

   the $\Gamma(\Pi)$ is said to be a set of *product weights*. We may refer to the set $\{\,\gamma_{d,j} : j\in[d], d\in\mathbb{N}\,\}$ as being the *weightlets* for $\Gamma(\Pi)$.

5. We say that $\Gamma(\text{FOW})$ is a family of *finite-order weights* if there exists $\omega \in \mathbb{N}_0$ such that
$$\gamma_{d,\mathfrak{u}} = 0 \qquad \text{for all } d \in \mathbb{N} \text{ and } \mathfrak{u} \text{ such that } |\mathfrak{u}| > \omega.$$
The smallest ω for which this holds is said to be the *order* of $\Gamma(\text{FOW})$. As a special case, we say that $\Gamma(\text{FDW})$ is a family of *finite-diameter weights* if
$$\gamma_{d,\mathfrak{u}} = 0 \qquad \text{for all } d \in \mathbb{N} \text{ and all } \mathfrak{u} \text{ with } \operatorname{diam}(\mathfrak{u}) \geq q.$$
The smallest q for which this holds is said to be the *diameter* of $\Gamma(\text{FDW})$. □

Remark 2. We can slightly simplify the sum appearing in (1), as in [12]. If we adopt the convention that $0/0 = 0$, we can write
$$\langle v, w\rangle_{H^{s,1}_\Gamma(\mathbb{T}^d)} = \sum_{\mathfrak{u}\subseteq[d]} \gamma^{-1}_{d,\mathfrak{u}} \langle \partial_\mathfrak{u} v, \partial_\mathfrak{u} w\rangle_{H^s(\mathbb{T}^d)} \qquad \forall\, v, w \in H^{s,1}_\Gamma(\mathbb{T}^d),$$
provided that we require
$$\partial_\mathfrak{u} w = 0 \text{ for any } \mathfrak{u} \subseteq [d] \text{ such that } \gamma_{d,\mathfrak{u}} = 0 \qquad \forall\, w \in H^{s,1}_\Gamma(\mathbb{T}^d).$$
Of course, if $\partial_\mathfrak{u} v = 0$, then $\partial_\mathfrak{v} w = 0$ for any superset $\mathfrak{v}$ of $\mathfrak{u}$. This imposes the natural condition
$$\gamma_{d,\mathfrak{u}} = 0 \implies \gamma_{d,\mathfrak{v}} = 0 \qquad \text{for any } \mathfrak{v} \subseteq [d] \text{ for which } \mathfrak{v} \supseteq \mathfrak{u}. \tag{4}$$
In the remainder of this paper, we shall assume that (4) *holds.* Now suppose that $\gamma_{d,j} = 0$ for some $j \in [d]$ and $d \in \mathbb{N}$. Using (4), we see that $\gamma_{d,\mathfrak{u}} = 0$ for any $\mathfrak{u}$ containing j as an element, and so the variable x_j plays no part in the problem $\text{App}_{d,\Gamma,r,s}$. So there is no essential loss of generality in assuming that
$$\gamma_{d,j} > 0 \qquad \forall\, j \in [d], d \in \mathbb{N}. \tag{5}$$
In the remainder of this paper, we shall also assume that (5) *holds for product weights.* □

An approximation is given by an algorithm $A_{d,\Gamma,r,s,n}$ using at most n linear functionals on $H^{s,1}_\Gamma(\mathbb{T}^d)$. That is, there exist continuous linear functionals $L_1, L_2 \dots, L_n$ on $H^{s,1}_\Gamma(\mathbb{T}^d)$ and a function $\phi_n \colon \mathbb{R}^n \to H^r(\mathbb{T}^d)$ such that
$$A_{d,\Gamma,r,s,n}(f) = \phi_n\left(L_1(f), L_2(f), \dots, L_n(f)\right) \qquad \forall\, f \in \mathscr{B}H^{s,1}_\Gamma(\mathbb{T}^d).$$
The worst case *error* of $A_{d,\Gamma,n,r,s}$ is given by
$$e(A_{d,\Gamma,r,s,n}) = \sup_{f\in\mathscr{B}H^{s,1}_\Gamma(\mathbb{T}^d)} \|f - A_{d,\Gamma,r,s,n} f\|_{H^r(\mathbb{T}^d)}.$$
For simplicity's sake, we measure the cost of an algorithm by the number of information evaluations it uses.

Let $\varepsilon > 0$ be a given error tolerance. An algorithm yields an *ε-approximation* if its error is at most ε. We define the *information complexity* $n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s})$ as the minimal number of linear functionals defined on $H^{s,1}_{\Gamma}(\mathbb{T}^d)$ needed to find an algorithm whose error as most ε.

As in [11], we have $\| \mathrm{App}_{d,\Gamma,r,s} \| = 1$. Hence it follows that

$$e(0, \mathrm{App}_{d,\Gamma,r,s}) = e(A_{d,r,s,0}, \mathrm{App}_{d,\Gamma,r,s}) = 1,$$

where $A_{d,r,s,0}$ is the *zero algorithm* defined by

$$A_{d,\Gamma,r,s,0} f \equiv 0 \qquad \forall\, f \in H^{s,1}_{\Gamma}(\mathbb{T}^d).$$

Thus $n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) = 0$ for $\varepsilon \geq 1$. So in the remainder of this paper, we assume that $\varepsilon \in (0,1)$, since the problem is trivial otherwise.

It is well-known that there exist algorithms with arbitrarily small error iff the operator $\mathrm{App}_{d,\Gamma,r,s}$ is compact. Since we want to find ε-approximations for any $\varepsilon \in (0,1)$, we shall assume that $\mathrm{App}_{d,\Gamma,r,s}$ is compact in the remainder of this paper. This compactness holds if either $r < s$ (by Rellich's Theorem, see e.g. [2, pg. 219]) or if at least one of the weights $\gamma_{d,\mathrm{u}}$ is positive (from the results in [11]).

Let $\{(\lambda_{d,n}, e_{d,n})\}_{n\in\mathbb{N}}$ denote the eigensystem of $W_{d,\Gamma,r,s} = \mathrm{App}^*_{d,\Gamma,r,s} \mathrm{App}_{d,\Gamma,r,s}$, with $H^{s,1}_{\Gamma}(\mathbb{T}^d)$-orthonormal eigenvectors $e_{d,n}$ and with the eigenvalues $\lambda_{d,n}$ forming a non-increasing sequence

$$\lambda_{d,1} = 1 \geq \lambda_{d,2} \geq \cdots > 0,$$

Then the algorithm

$$A_n(f) = \sum_{i=1}^{n} \langle f, e_{d,i} \rangle_{H^{s,1}_{\Gamma}(\mathbb{T}^d)} e_{d,i} = \sum_{i=1}^{n} \lambda_{d,i}^{-1} \langle f, e_{d,i} \rangle_{L_2(\mathbb{T}^d)} e_{d,i} \qquad \forall\, f \in \mathscr{B}H^{s,1}_{\Gamma}(\mathbb{T}^d)$$

minimizes the worst case error among *all* algorithms using n linear functionals on $H^{s,1}_{\Gamma}(\mathbb{T}^d)$, with error

$$e(A_n) = \sqrt{\lambda_{d,n+1}},$$

so that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) = \inf\{\, n \in \mathbb{N}_0 : \lambda_{d,n} > \varepsilon^2 \,\}. \tag{6}$$

We are now ready to describe various levels of tractability for the approximation problem $\mathrm{App}_{\Gamma,r,s} = \{\mathrm{App}_{d,\Gamma,r,s}\}_{d\in\mathbb{N}}$. This problem can satisfy any of the following tractability criteria, listed in decreasing order of desirability, see [7] for further discussion.

1. The problem is *strongly* (polynomial) *tractable* if there exists $p \geq 0$ such that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \leq C \left(\frac{1}{\varepsilon}\right)^p \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{N}. \tag{7}$$

When this holds, we define

$$p(\mathrm{App}_{\Gamma,r,s}) = \inf\{\, p \geq 0 : (7) \text{ holds}\}$$

to be the *exponent of strong tractability*.

2. The problem is (polynomially) *tractable* if there exist non-negative numbers C, p, and q such that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \leq C\left(\frac{1}{\varepsilon}\right)^p d^q \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{N}. \tag{8}$$

Numbers $p = p(\mathrm{App}_{\Gamma,r,s})$ and $q = q(\mathrm{App}_{\Gamma,r,s})$ such that (8) holds are called ε- and d-*exponents of tractability*; these need not be uniquely defined.

3. The problem is *quasi-polynomially tractable* if there exist $C \geq 0$ and $t \geq 0$ such that

$$n(\varepsilon, S_d) \leq C \exp\bigl(t(1+\ln \varepsilon^{-1})(1+\ln d)\bigr) \qquad \forall\, \varepsilon \in (0,1), \forall\, d \in \mathbb{N}. \tag{9}$$

The infimum of all t such that (9) holds is said to be the *exponent of quasi-polynomial tractability*, denoted t^{qpoly}.

4. Let t_1 and t_2 be non-negative numbers. The problem is (t_1,t_2)-*weakly tractable* if non-negative numbers, with

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s})}{\varepsilon^{-t_1} + d^{t_2}} > 0. \tag{10}$$

The problem is said to be *weakly tractable* if it is $(1,1)$-weakly tractable, and *uniformly weakly tractable* if it is (t_1,t_2)-weakly tractable for all positive t_1 and t_2. For more details, see [10].

5. The problem is *intractable* if it is not (t_1,t_2)-weakly tractable for any non-negative t_1 and t_2.

6. The problem suffers from the *curse of dimensionality* if there exists $c > 1$ such that[3]

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \geq c^d \qquad \forall\, d \in \mathbb{N}. \tag{11}$$

3 Spectral Results

If we want to follow the prescription for determining minimal error algorithms for our problem, we clearly need to know the eigenvalues and eigenvectors of $W_{d,\Gamma,r,s}$. That's what we'll be doing in this section.

First, a bit more notation. Let $\mathrm{i} = \sqrt{-1}$. For $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{T}^d$, let $\mathbf{k}\cdot\boldsymbol{x} = \sum_{j=1}^d k_j x_j$. Define

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[3] We follow [5, (5.3)] in using $1+c$ with $c > 0$ rather than $c > 1$.

$$e_{d,\mathbf{k}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}} \exp(\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \mathbb{T}^d.$$

For any $f \in H^r(\mathbb{T}^d)$, we have

$$D^{\mathbf{m}} f = \sum_{\mathbf{k}\in\mathbb{Z}^d} (\mathrm{i}\,\mathbf{k})^{\mathbf{m}} c_{d,\mathbf{k}}(f)\, e_{d,\mathbf{k}} \qquad \text{for } |\mathbf{m}| \le r,$$

where

$$c_{d,\mathbf{k}}(f) = \int_{\mathbb{T}^d} f(\boldsymbol{x}) \exp(-\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

is the $\mathbf{k}$th Fourier coefficient of f and convergence is in the $L_2(\mathbb{T}^d)$-sense.

Theorem 1. *For $\mathbf{k} \in \mathbb{Z}^d$, let*

$$\lambda_{d,\mathbf{k},\Gamma,r,s} = \frac{\beta_{d,r,\mathbf{k}}}{\beta_{d,s,\mathbf{k}}}\, \alpha_{d,\mathbf{k},\Gamma}, \tag{12}$$

where

$$\alpha_{d,\mathbf{k},\Gamma} = \Bigg(\sum_{\substack{\mathfrak{u}\subseteq[d] \\ \gamma_{d,\mathfrak{u}}>0}} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} k_j^2 \Bigg)^{-1} \tag{13}$$

and

$$\beta_{d,r,\mathbf{k}} = \sum_{|\mathbf{m}|\le r} \mathbf{k}^{2\mathbf{m}}. \tag{14}$$

Then the following hold:

1. *The vectors $\{e_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ form an orthogonal basis for $H^{s,1}_{\Gamma}(\mathbb{T}^d)$, with*

$$\|e_{d,\mathbf{k}}\|^2_{H^{s,1}_{\Gamma}(\mathbb{T}^d)} = \alpha^{-1}_{d,\mathbf{k},\Gamma}\, \beta_{d,s,\mathbf{k}} \qquad \forall \mathbf{k} \in \mathbb{Z}^d. \tag{15}$$

2. *The vectors $\{e_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ form an orthogonal basis for $H^r(\mathbb{T}^d)$, with*

$$\|e_{d,\mathbf{k}}\|^2_{H^r(\mathbb{T}^d)} = \beta_{d,r,\mathbf{k}} \qquad \forall \mathbf{k} \in \mathbb{Z}^d. \tag{16}$$

3. *The eigensystem of $W_{d,\Gamma,r,s}$ is given by $\big\{ \big(\lambda_{d,\mathbf{k},\Gamma,r,s}, e_{d,\mathbf{k}}\big) \big\}_{\mathbf{k}\in\mathbb{Z}^d}$, so that*

$$W_{d,\Gamma,r,s} e_{d,\mathbf{k}} = \lambda_{d,\mathbf{k},\Gamma,r,s}\, e_{d,\mathbf{k}} \qquad \forall \mathbf{k} \in \mathbb{Z}^d. \tag{17}$$

4. *The information complexity is given by*

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) = \Big| \big\{ \mathbf{k} \in \mathbb{Z}^d : \lambda_{d,\mathbf{k},\Gamma,r,s} > \varepsilon^2 \big\} \Big|. \tag{18}$$

Proof. For part 1, we need to show that that $\{e_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ is an orthogonal basis for $H^{s,1}_{\Gamma}(\mathbb{T}^d)$. Let $v \in H^{s,1}_{\Gamma}(\mathbb{T}^d)$. Now for any $\mathbf{k} \in \mathbb{Z}^d$ and any $\mathfrak{u} \subseteq [d]$, we have

$$\partial_{\mathfrak{u}} D^{\mathbf{m}} e_{d,\mathbf{k}} = \prod_{j\in\mathfrak{u}} (-\mathrm{i}\, k_j) \prod_{j=1}^{d} (-\mathrm{i}\, k_j)^{m_j}\, e_{d,\mathbf{k}} = (-\mathrm{i})^{|\mathfrak{u}|+|\mathbf{m}|} \mathbf{k}^{\mathbf{m}} e_{d,\mathbf{k}}$$

and so we may integrate by parts and use periodicity to see that

$$\begin{aligned}\langle \partial_{\mathfrak{u}} D^{\mathbf{m}} v, \partial_{\mathfrak{u}} D^{\mathbf{m}} e_{d,\mathbf{k}} \rangle_{L_2(\mathbb{T}^d)} &= (-1)^{|\mathfrak{u}|+|\mathbf{m}|} \langle v, \partial_{\mathfrak{u}}^2 D^{2\mathbf{m}} e_{d,\mathbf{k}} \rangle_{L_2(\mathbb{T}^d)} \\ &= (-1)^{|\mathfrak{u}|+|\mathbf{m}|} (-\mathrm{i})^{2(|\mathfrak{u}|+\mathbf{m}|)} \left(\prod_{j\in\mathfrak{u}} k_j^2 \right) \mathbf{k}^{2|\mathbf{m}|} \langle v, e_{d,\mathbf{k}} \rangle_{L_2(\mathbb{T}^d)} \\ &= \left(\prod_{j\in\mathfrak{u}} k_j^2 \right) \mathbf{k}^{2\mathbf{m}} \langle v, e_{d,\mathbf{k}} \rangle_{L_2(\mathbb{T}^d)}.\end{aligned}$$

Hence for any $\mathbf{k} \in \mathbb{Z}^d$, we have

$$\begin{aligned}\langle v, e_{d,\mathbf{k}} \rangle_{H^{s,1}_{\Gamma}(\mathbb{T}^d)} &= \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}>0}} \gamma_{d,\mathfrak{u}}^{-1} \langle \partial_{\mathfrak{u}} v, \partial_{\mathfrak{u}} e_{d,\mathbf{k}} \rangle_{H^s(\mathbb{T}^d)} \\ &= \sum_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}>0}} \gamma_{d,\mathfrak{u}}^{-1} \sum_{|\mathbf{m}|\le s} \langle \partial_{\mathfrak{u}} D^{\mathbf{m}} v, \partial_{\mathfrak{u}} D^{\mathbf{m}} e_{d,\mathbf{k}} \rangle_{L_2(\mathbb{T}^d)} \\ &= \left(\sum_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}>0}} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} k_j^2 \right) \left(\sum_{|\mathbf{m}|\le s} \mathbf{k}^{2|\mathbf{m}|} \right) \langle v, e_{d,\mathbf{k}} \rangle_{L_2(\mathbb{T}^d)} \\ &= \alpha_{d,\mathbf{k},\Gamma}^{-1}\, \beta_{d,s,\mathbf{k}} \langle v, e_{d,\mathbf{k}} \rangle_{L_2(\mathbb{T}^d)}.\end{aligned} \tag{19}$$

In particular, we see that

$$\langle e_{d,\mathbf{p}}, e_{d,\mathbf{k}} \rangle_{H^{s,1}_{\Gamma}(\mathbb{T}^d)} = \alpha_{d,\mathbf{k},\Gamma}^{-1}\, \beta_{d,s,\mathbf{k}}\, \delta_{\mathbf{p},\mathbf{k}} \qquad \forall \mathbf{k}, \mathbf{p} \in \mathbb{Z}^d, \tag{20}$$

with $\delta_{\mathbf{k},\mathbf{p}}$ being the Kronecker delta. Hence $\{e_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ is an $H^{s,1}_{\Gamma}(\mathbb{T}^d)$-orthogonal set, the norm of whose elements being given by (15). To see that this set is a basis, we need only show that this set is $H^{s,1}_{\Gamma}(\mathbb{T}^d)$-complete. So let $v \in H^{s,1}_{\Gamma}(\mathbb{T}^d)$ satisfy $\langle v, e_{d,\mathbf{k}} \rangle_{H^{s,1}_{\Gamma}(\mathbb{T}^d)} = 0$ for all $\mathbf{k} \in \mathbb{Z}^d$. Once again using (19), it follows that $\langle v, e_{d,\mathbf{k}} \rangle_{L_2(\mathbb{T}^d)} = 0$ for all $\mathbf{k} \in \mathbb{Z}^d$. Since $\{e_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ is an orthogonal basis for $L_2(\mathbb{T}^d)$, it follows that $v = 0$. Hence $\{e_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ is $H^{s,1}_{\Gamma}(\mathbb{T}^d)$-complete, as required.

Setting $\Gamma = \Gamma(\emptyset)$ in part 1, we immediately have part 2.

To see that part 3 holds, note that

$$\langle W_{d,\Gamma,r,s} e_{d,\mathbf{k}}, e_{d,\mathbf{p}} \rangle_{H^{s,1}_{\Gamma}(\mathbb{T}^d)} = \langle e_{d,\mathbf{k}}, e_{d,\mathbf{p}} \rangle_{H^r(\mathbb{T}^d)} = \beta_{d,r,\mathbf{k}}\, \delta_{\mathbf{k},\mathbf{p}}, \qquad \forall \mathbf{k}, \mathbf{p} \in \mathbb{Z}^d,$$

the second equality following from part 2. Since $\{e_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ is an orthogonal basis for $H^{s,1}_{\Gamma}(\mathbb{T}^d)$, it follows that $W_{d,\Gamma,r,s} e_{d,\mathbf{k}}$ must be a multiple of $e_{d,\mathbf{k}}$, which means that $e_{d,\mathbf{k}}$ is an eigenvector of $W_{d,\Gamma,r,s}$. Thus $W_{d,\Gamma,r,s} e_{d,\mathbf{k}} = \lambda_{d,\mathbf{k},\Gamma,r,s} e_{d,\mathbf{k}}$ for some $\lambda_{d,\mathbf{k},\Gamma,r,s} > 0$, with

$$\lambda_{d,\mathbf{k},\Gamma,r,s} = \frac{\|e_{d,\mathbf{k}}\|^2_{H^r(\mathbb{T}^d)}}{\|e_{d,\mathbf{k}}\|^2_{H^{s,1}_\Gamma(\mathbb{T}^d)}}, \tag{21}$$

as usual. Part 3 follows once we use (15) and (16) in (21).

Finally, part 4 follows immediately from (6), along with the remaining parts of this theorem. □

As a special case, let $s = r$. Then the problem $\mathrm{App}_{d,\Gamma,r,r}$ is equivalent to the problem $\mathrm{App}_{d,\Gamma,0,0}$:

Corollary 1. *The following results hold for the problem* $\mathrm{App}_{d,\Gamma,r,r}$*:*

1. *The operators $W_{d,\Gamma,r,r}$ and $W_{d,\Gamma,0,0}$ both have $\{(e_{d,\mathbf{k}}, \alpha_{d,\mathbf{k},\Gamma})\}_{\mathbf{k}\in\mathbb{Z}^d}$ as their eigensystems.*
2. *Minimal errors, minimal error algorithms, and levels of tractability are the same for our problem $\mathrm{App}_{d,\Gamma,r,r}$ and for the problem $\mathrm{App}_{d,\Gamma,0,0}$.* □

Just as we have reduced the problem $\mathrm{App}_{d,\Gamma,r,r}$ to the problem $\mathrm{App}_{d,\Gamma,0,0}$, we can also reduce the problem $\mathrm{App}_{d,\Gamma,r,s}$ to the simpler problem $\mathrm{App}_{d,\Gamma,0,s-r}$. Let

$$\eta_{d,\mathbf{k}} = 1 + \sum_{j=1}^{d} k_j^2. \tag{22}$$

We then have

Theorem 2. *Let $r, s \in \mathbb{N}_0$, with $s \geq r$.*

1. *The eigenvectors of $W_{d,\Gamma,r,s}$ are given by $\{\, e_{d,\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d \,\}$.*
2. *The eigenvalues of $W_{d,\Gamma,r,s}$ satisfy the inequality*

$$\frac{1}{r!(s-r)!}\,\lambda_{d,\mathbf{k},\Gamma,0,s-r} \leq \frac{1}{r!}\,\frac{\alpha_{d,\mathbf{k},\Gamma}}{\eta_{d,\mathbf{k}}^{s-r}} \leq \lambda_{d,\mathbf{k},\Gamma,r,s} \leq s!\,\frac{\alpha_{d,\mathbf{k},\Gamma}}{\eta_{d,\mathbf{k}}^{s-r}} \leq s!\,\lambda_{d,\mathbf{k},\Gamma,0,s-r} \tag{23}$$

for all $\mathbf{k} \in \mathbb{Z}^d$.

Proof. Let $d \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{Z}^d$. As in [5], we may use the multinomial theorem to see that

$$\beta_{d,\ell,\mathbf{k}} \leq \eta_{d,\mathbf{k}}^{\ell} \leq \ell!\,\beta_{d,\ell,\mathbf{k}}. \qquad \forall \ell \in \mathbb{N}_0.$$

We then have

$$\frac{1}{r!\,(s-r)!\,\beta_{d,s-r,\mathbf{k}}} \leq \frac{1}{r!\,\eta_{d,\mathbf{k}}^{s-r}} \leq \frac{\beta_{d,r,\mathbf{k}}}{\beta_{d,s,\mathbf{k}}} \leq \frac{s!}{\eta_{d,\mathbf{k}}^{s-r}} \leq \frac{s!}{\beta_{d,s-r,\mathbf{k}}}.$$

This result now follows from Theorem 1 and (18). □

From Theorem 2, we see that minimal errors for our problem $\mathrm{App}_{d,\Gamma,r,s}$ and for the simpler problem $\mathrm{App}_{d,\Gamma,0,s-r}$ are essentially the same.

4 Tractability Results

We now compare the tractability of our problem $\mathrm{App}_{\Gamma,r,s} = \{\mathrm{App}_{d,\Gamma,r,s}\}_{d\in\mathbb{N}}$ with the problem $\mathrm{App}_{\Gamma,0,0} = \{\mathrm{App}_{d,\Gamma,0,0}\}_{d\in\mathbb{N}}$. The papers [11, 12] studied this latter problem, except for functions defined over the unit cube instead of the unit torus.

4.1 General Weights

We first give tractability results that hold for any weights, regardless of their structure (or lack thereof), depending only some boundedness conditions. Our main result is that our approximation problem $\mathrm{App}_{\Gamma,r,s}$ has the same level of tractability as the problem $\mathrm{App}_{\Gamma,0,0}$, which is the periodic version of the problem studied in [12]. In what follows, we let

$$M_d = \max\left\{1, \max_{j\in[d]} \gamma_{d,\{j\}}\right\} \qquad \text{and} \qquad m_d = \min_{\substack{\mathfrak{u}\subseteq[d]\\ \gamma_{d,\mathfrak{u}}>0}} \gamma_{d,\mathfrak{u}}. \tag{24}$$

Clearly both M_d and m_d are positive numbers.

First, we compare the information complexity of these problems.

Theorem 3. *For all $d\in\mathbb{N}$ and $\varepsilon\in(0,1)$, we have*

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \ge n\left(\left(r!M_d^{s-r}\right)^{1/(2(s-r+1))} \varepsilon^{1/(s-r+1)}, \mathrm{App}_{d,\Gamma,0,0}\right), \tag{25}$$

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \le n\left(\left(\frac{m_d^{s-r}}{s!}\right)^{1/(2(s-r+1))} \varepsilon^{1/(s-r+1)}, \mathrm{App}_{d,\Gamma,0,0}\right), \tag{26}$$

and

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \le n(\varepsilon, \mathrm{App}_{d,\Gamma,0,0}). \tag{27}$$

Proof. We first show that (25) holds. Let $\mathbf{k}\in\mathbb{Z}^d$. From (5), (13), (18), and (22), it follows that

$$\alpha_{d,\mathbf{k},\Gamma}^{-1} \ge 1 + \sum_{j=1}^d \gamma_{d,j}^{-1} k_j^2 \ge 1 + \min_{j\in[d]} \gamma_{d,j}^{-1} \sum_{j=1}^d k_j^2 \ge M_d^{-1}\left(1 + \sum_{j=1}^d k_j^2\right) = M_d^{-1}\,\eta_{d,\mathbf{k}}.$$

Since $\eta_{d,\mathbf{k}}^{-1} \ge M_d^{-1}\alpha_{d,\mathbf{k},\Gamma}$, we may use Theorem 2 to see that

$$\lambda_{d,\mathbf{k},\Gamma,r,s} \ge \frac{1}{r!}\frac{\alpha_{d,\mathbf{k},\Gamma}}{\eta_{d,\mathbf{k}}^{s-r}} \ge \frac{1}{r!M_d^{s-r}}\alpha_{d,\mathbf{k},\Gamma}^{s-r+1}.$$

Using part 4 of Theorem 1 and the previous estimate, we now have

$$
\begin{aligned}
n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) &= \left|\{\, \mathbf{k} \in \mathbb{Z}^d : \lambda_{d,\mathbf{k},\Gamma,r,s} > \varepsilon^2 \,\}\right| \\
&\geq \left|\left\{ \mathbf{k} \in \mathbb{Z}^d : \frac{1}{r! M_d^{s-r}} \alpha_{d,\mathbf{k},\Gamma}^{s-r+1} > \varepsilon^2 \right\}\right| \\
&= \left|\left\{ \mathbf{k} \in \mathbb{Z}^d : \alpha_{d,\mathbf{k},\Gamma} > \left(r! M_d^{s-r} \varepsilon^2\right)^{1/(s-r+1)} \right\}\right| \\
&= n\left(\left(r! M_d^{s-r}\right)^{1/(2(s-r+1))} \varepsilon^{1/(s-r+1)}, \mathrm{App}_{d,\Gamma,0,0} \right),
\end{aligned}
$$

as required.

The proof of (26) is similar to that of (25), except that we start with the bound

$$
\alpha_{d,\mathbf{k},\Gamma}^{-1} = \sum_{\substack{\mathfrak{u} \subseteq [d] \\ \gamma_{d,\mathfrak{u}} > 0}} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j \in \mathfrak{u}} k_j^2 \leq m_d^{-1} \sum_{\substack{\mathfrak{u} \subseteq [d] \\ \gamma_{d,\mathfrak{u}} > 0}} \prod_{j \in \mathfrak{u}} k_j^2 \leq m_d^{-1} \beta_{d,1,\mathbf{k}} = m_d^{-1} \eta_{d,k}.
$$

Finally, (27) follows from (18) and Theorem 1. □

We now show that the level of tractability of our problem $\mathrm{App}_{\Gamma,r,s}$ is often the same as that of the problem $\mathrm{App}_{\Gamma,0,0}$.

Theorem 4. *If* $\mathrm{App}_{\Gamma,0,0}$ *has a given level of tractability, then* $\mathrm{App}_{\Gamma,r,s}$ *has at least the same level of tractability, and the exponent(s) for* $\mathrm{App}_{\Gamma,r,s}$ *are bounded from above by those for* $\mathrm{App}_{\Gamma,0,0}$*. Moreover, recalling the defintion* (24) *of* M_d*, we have the following:*

1. If

$$
M := \sup_{d \in \mathbb{N}} M_d < \infty \tag{28}
$$

then the following hold:

a. $\mathrm{App}_{\Gamma,r,s}$ *is strongly polynomially tractable iff* $\mathrm{App}_{\Gamma,0,0}$ *is strongly polynomially tractable, in which case the exponents of strong tractability satisfy the inequality*

$$
\frac{1}{s-r+1}\, p(\mathrm{App}_{\Gamma,0,0}) \leq p(\mathrm{App}_{\Gamma,r,s}) \leq p(\mathrm{App}_{\Gamma,0,0}). \tag{29}
$$

b. $\mathrm{App}_{\Gamma,r,s}$ *is quasi-polynomially tractable iff* $\mathrm{App}_{\Gamma,0,0}$ *is quasi-polynomially tractable, in which case the exponents of strong quasi-polynomial tractability satisfy the inequality*

$$
\begin{aligned}
\frac{1}{\max\left\{s-r, \frac{1}{2}\ln\left(r! M^{s-r}\right)\right\}+1}\, t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,0,0}) &\leq t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,r,s}) \\
&\leq t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,0,0}).
\end{aligned} \tag{30}
$$

2. If

$$
\sup_{d \in \mathbb{N}} d^{-q} M_d < \infty \tag{31}
$$

for some $q \geq 0$, then $\mathrm{App}_{\Gamma,r,s}$ *is polynomially tractable iff* $\mathrm{App}_{\Gamma,0,0}$ *is polynomially tractable.*

Proof. The first statement in the theorem follows immediately from (27).

For part 1, suppose that (28) holds.

We first prove part 1(a). From the first statement in the theorem, it suffices to show that if $\mathrm{App}_{\Gamma,r,s}$ is strongly polynomially tractable, then the same is true for $\mathrm{App}_{\Gamma,0,0}$, and that the first inequality in (29) holds. So let $\mathrm{App}_{\Gamma,r,s}$ ben strongly polynomially tractable, so that for any $p > p(\mathrm{App}_{\Gamma,r,s})$, there exists $C > 0$ such that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \leq C\varepsilon^{-p} \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{N}.$$

Set

$$\varepsilon_d = \left(r! M_d^{s-r}\right)^{1/(2(s-r+1))} \varepsilon^{1/(s-r+1)}, \tag{32}$$

so that

$$\varepsilon^{-1} = (r! M_d^{s-r})^{1/2} \varepsilon_d^{-(s-r+1)}.$$

Using (25) and (32), we see that

$$\begin{aligned} n(\varepsilon_d, \mathrm{App}_{d,\Gamma,0,0}) \leq n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) &\leq C\varepsilon^{-p} = C\left(M_d^{s-r} r!\right)^{p/2} \varepsilon_d^{-(s-r+1)p} \\ &\leq C\left(M^{s-r} r!\right)^{p/2} \varepsilon_d^{-(s-r+1)p}. \end{aligned}$$

Varying $\varepsilon > 0$, we see that ε_d can assume arbitrary positive values here. Since p may be chosen arbitrarily close to $p(\mathrm{App}_{\Gamma,r,s})$, we see that $\mathrm{App}_{\Gamma,0,0}$ is strongly polynomially tractable, and that (29) holds, as required.

We now prove part 1(b). It suffices to show that if $\mathrm{App}_{\Gamma,r,s}$ is strongly quasi-polynomially tractable, then so is $\mathrm{App}_{\Gamma,0,0}$, and that the first inequality in (30) holds. So suppose that $\mathrm{App}_{\Gamma,0,0}$ is quasi-polynomially tractable. Then for any $t > t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,0,0})$, there exists $C > 0$ such that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \leq C \exp\left(t(1+\ln\varepsilon^{-1})(1+\ln d)\right) \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{N}.$$

Once again, define ε_d by (32) and use (25) to see that

$$\begin{aligned} n(\varepsilon_d, \mathrm{App}_{d,\Gamma,0,0}) &\leq n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \leq C \exp\left(t(1+\ln\varepsilon^{-1})(1+\ln d)\right) \\ &= C \exp\left[t\left(1+(s-r+1)\ln\varepsilon_d^{-1} + \tfrac{1}{2}\ln(r! M_d^{s-r})\right)(1+\ln d)\right] \\ &\leq C \exp\left[t\left(1+(s-r+1)\ln\varepsilon_d^{-1} + \tfrac{1}{2}\ln(r! M^{s-r})\right)(1+\ln d)\right] \end{aligned} \tag{33}$$

Define $g\colon [0,\infty) \to [0,\infty)$ as

$$g(\xi) = \frac{1+(s-r+1)\xi + \frac{1}{2}\ln(r! M^{s-r})}{1+\xi} \qquad \forall\, \xi \geq 0.$$

We find that

$$\sup_{\xi\geq 0} g(\xi) = \max\left\{g(0), \lim_{\xi\to\infty} g(\xi)\right\} = \max\left\{s-r, \tfrac{1}{2}\ln(r!M^{s-r})\right\} + 1.$$

From (33), we now see that

$$n(\varepsilon_d, \mathrm{App}_{d,\Gamma,0,0}) \leq C\exp\left(t_1(1+\ln\varepsilon_d^{-1})(1+\ln d)\right),$$

where

$$t_1 = t\sup_{d\in\mathbb{N}} g(\ln\varepsilon_d^{-1}) = t\left(\max\left\{s-r, \tfrac{1}{2}\ln(r!M^{s-r})\right\}+1\right). \tag{34}$$

Arguing as in the strongly polynomial case, we see that $\mathrm{App}_{\Gamma,0,0}$ is quasi-polynomially tractable, with

$$t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,r,s}) \leq \left(\max\left\{s-r, \tfrac{1}{2}\ln(r!M^{s-r})\right\}+1\right) t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,0,0}),$$

as required.

For part 2, suppose that (31) holds, so that $M := \sup_{d\in\mathbb{N}} d^{-q}M_d < \infty$. Suppose also that $\mathrm{App}_{\Gamma,r,s}$ is polynomially tractable, so that there exist positive C, ℓ, and p such that such that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,0,0}) \leq Cd^{\ell}\varepsilon^{-p} \qquad \forall d\in\mathbb{N}, \varepsilon\in(0,1).$$

Once again defining ε_d as in (32), we have

$$\begin{aligned} n(\varepsilon_d, \mathrm{App}_{d,\Gamma,0,0}) &\leq Cd^{\ell}\varepsilon^{-p} = Cd^{\ell}(r!M_d^{s-r})^{p/2}\varepsilon_d^{-(s-r+1)p} \\ &\leq Cd^{\ell}(r!M^{s-r})^{p/2}\varepsilon_d^{-(s-r+1)p}. \end{aligned}$$

Hence $\mathrm{App}_{\Gamma,0,0}$ is polynomially tractable. □

Remark 3. The non-trivial results in Theorem 4 hold when the boundedness conditions (28) or (31) are satisfied. Suppose that we allow unbounded weights. Although the tractability of $\mathrm{App}_{\Gamma,r,s}$ is no worse than the tractability of $\mathrm{App}_{\Gamma,0,0}$, we can say nothing in the opposite direction in this case. As an extreme example, we show a choice of (unbounded) weights such that $\mathrm{App}_{\Gamma,r,s}$ to be strongly polynomially tractable, but for which $\mathrm{App}_{\Gamma,0,0}$ suffers from the curse of dimensionality.

Define our weight set Γ as

$$\gamma_{d,\mathfrak{u}} = \begin{cases} 1 & \text{if } \mathfrak{u}=\emptyset, \\ (1+c)^{2d} & \text{if } \mathfrak{u}=\{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

This is actually a sequence of univariate problems, for which

$$\alpha_{1,k,\Gamma} = \left(1+(1+c)^{-2d}\right)k^2 \qquad \text{and} \qquad \eta_{1,k} = 1+k^2.$$

From Theorem 2, we see that the eigenvalues of $W_{1,\Gamma,r,s}$ satisfy

$$\lambda_{1,k} \le s!\,\frac{\alpha_{1,k,\Gamma}}{\eta_{1,k}^{s-r}} = \frac{s!}{\bigl(1+(1+c)^{-2d}(1+k^2)\bigr)(1+k^2)^{s-r}} \le \frac{s!}{(1+k^2)^{s-r}}.$$

Hence we may use (23) to see that

$$\begin{aligned} n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) &\le \left| \left\{ k \in \mathbb{Z} : \frac{s!}{(1+k^2)^{s-r}} > \varepsilon^2 \right\} \right| = 2 \left\lfloor \sqrt{\left(\frac{s!}{\varepsilon^2}\right) - 1} \right\rfloor - 1 \\ &= \Theta\left(\varepsilon^{1/(s-r)}\right), \end{aligned}$$

and so $\mathrm{App}_{\Gamma,r,s}$ is strongly polynomially tractable, provided that $r < s$. On the other hand, we have

$$\begin{aligned} n(\varepsilon, \mathrm{App}_{d,\Gamma,0,0})` &= |\{ k \in \mathbb{Z} : \alpha_{1,k,\Gamma} > \varepsilon^2 \}| = |\{ k \in \mathbb{Z} : 1 + (1+c)^{-2d} k^2 > \varepsilon^2 \}| \\ &= 2 \left\lfloor (1+c)^d \sqrt{\varepsilon^{-2} - 1} \right\rfloor - 1 = \Theta\left((1+c)^d \varepsilon^{-1}\right), \end{aligned}$$

and so $\mathrm{App}_{\Gamma,0,0}$ suffers from the curse of dimensionality. □

Remark 4. If we are willing to live with an upper bound that depends on d, we can improve the ε-exponent in Theorem 4. (This is an example of the tradeoff of exponents, as described several places in [7].) To be specific, suppose that $\mathrm{App}_{\Gamma,0,0}$ is strongly polynomially tractable,. Then for any $p > p(\mathrm{App}_{\Gamma,0,0})$, there exists $C > 0$ such that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,0,0}) \le C\varepsilon^{-p} \qquad \forall \varepsilon \in (0,1), d \in \mathbb{N}.$$

Choosing such a p, d, and ε, let

$$\varepsilon_d = \left(\frac{m_d^{s-r}}{s!}\right)^{1/(2(s-r+1))} \varepsilon^{1/(s-r+1)},$$

where m_d is defined by (24). Using (26), the previous inequality tells us that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \le n(\varepsilon_d, \mathrm{App}_{\Gamma,0,0}) \le C \left(\frac{s!}{m_d^{s-r}}\right)^{p/(2(s-r+1))} \varepsilon^{-p/(s-r+1)}. \qquad (35)$$

Let $m = \inf_{d\in\mathbb{N}} m_d$. There are two cases to consider:

1. Suppose that $m > 0$. Then the $H_\Gamma^{s,1}(\mathbb{T}^d)$-norms are equivalent to the $H^{s,1}(\mathbb{T}^d)$-norms, with equivalence factors independent of d. As we shall see in Section 4.2, the problems $\mathrm{App}_{\Gamma(\mathrm{UNW}),r,s}$ and $\mathrm{App}_{\Gamma(\mathrm{UNW}),0,0}$ are both quasi-polynomially tractable, each having exponent $2/\ln 2 \doteq 2.88539$. Hence the same is true for the problems $\mathrm{App}_{\Gamma,r,s}$ and $\mathrm{App}_{\Gamma,0,0}$. Thus part 1(a) of Theorem 4 never comes into play when $m > 0$, and so the estimate (35) does not apply.
2. Suppose that $m = 0$. Then the bound (35) truly depends on d. To cite two examples:
   - Suppose that $m_d \ge C_\alpha d^{-\alpha}$ for some $\alpha > 0$ and $C_\alpha > 0$. Using (35), and letting

$$C_1 = C\left(\frac{s!}{C_\alpha^{s-r}}\right)^{p/(2(s-r+1))},$$

we see that

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \le C_1\, d^{\alpha p(s-r)/(2(s-r+1))}\, \varepsilon^{-p/(s-r+1)}.$$

Since p can be chosen arbitrarily close to $p(\mathrm{App}_{\Gamma,0,0})$, this is a polynomially-tractable upper bound on $n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s})$, with

$$d\text{-exponent: } \frac{\alpha(s-r)\,p(\mathrm{App}_{\Gamma,0,0})}{2(s-r+1)} \qquad \text{and} \qquad \varepsilon^{-1}\text{-exponent: } \frac{p(\mathrm{App}_{\Gamma,0,0})}{s-r+1}.$$

- Suppose that for *any* $\alpha > 0$, there exists $C_\alpha >$ such that $m_d \ge C_\alpha d^{-\alpha}$. (For instance, this holds if m_d is bounded from below by a power of $\log d$.) We now see that the results of the previous case hold for positive α, no matter how small. Hence we find that $\mathrm{App}_{\Gamma,r,s}$ is polynomially tractable for such Γ, with

$$d\text{-exponent: } 0 \qquad \text{and} \qquad \varepsilon^{-1}\text{-exponent: } \frac{1}{s-r+1}\,p(\mathrm{App}_{\Gamma,0,0}).$$

This is close to, but not identical to, a strong polynomial bound for which

$$p(\mathrm{App}_{\Gamma,r,s}) = \frac{1}{s-r+1}\,p(\mathrm{App}_{\Gamma,0,0}). \tag{36}$$

We might describe such a bound as being *almost strongly polynomial*. □

Remark 5. From Remark 4 we see that the left-hand inequality in (29) cannot be improved. However, this fact does not imply that there are problems for which (36) holds. To see that such problems do exist, suppose we choose our weights as

$$\gamma_{d,\mathfrak{u}} = \begin{cases} 1 & \text{if } \mathfrak{u} = \emptyset \text{ or } \mathfrak{u} = \{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that (36) holds for this problem. Indeed, the eigenvalues of $W_{d,\Gamma,0,0}$ are given by $1/(1+k^2)$ for $k \in \mathbb{Z}$, so that Theorem 2 tells us that the eigenvalues of $W_{d,\Gamma,r,s}$ are bounded from below by $1/\big(r!(1+k^2)^{s-r+1}\big)$ and from above by $s!/(1+k^2)^{s-r+1}$. It now follows that $n(\varepsilon, \mathrm{App}_{\Gamma,0,0}) = \Theta(\varepsilon^{-1})$ and $n(\varepsilon, \mathrm{App}_{\Gamma,r,s}) = \Theta(\varepsilon^{-1/(s-r+1)})$. Since $p(\mathrm{App}_{\Gamma,0,0}) = 1$ and $p(\mathrm{App}_{\Gamma,r,s}) = 1/(s-r+1)$, we see that (36) holds, as claimed. □

Remark 6. Note that Theorem 4 doesn't mention (r_1, r_2)-weak tractability. That's because (r_1, r_2)-weak tractability simply never arises. To see this, we distinguish between two cases:

1. Suppose that we allow Γ to contain an unbounded sequence of weights. Using Remark 3, we can find a case in which $\mathrm{App}_{\Gamma,0,0}$ suffers from the curse of dimensionality, but $\mathrm{App}_{\Gamma,r,s}$ is strongly polynomially tractable.
2. The alternative is to suppose that the weights are uniformly bounded, with $M = \sup_{d\in\mathbb{N}} \max_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} < \infty$. We claim that $\mathrm{App}_{\Gamma,0,0}$ is always (at least) quasi-polynomially tractable in this case, so that the same is true for $\mathrm{App}_{\Gamma,r,s}$ by part 1(b) of Theorem 4.
   Indeed, to see that $\mathrm{App}_{\Gamma,0,0}$ with weights bounded by M is always (at least) quasi-polynomially tractable, note that this problem is no harder than the problem $\mathrm{App}_{\Gamma,0,0}$ for which $\gamma_{d,\mathfrak{u}} \equiv M$. From Theorem 1, we see that the eigenvalues of this latter problem are given by
   $$\lambda_{d,\mathbf{k},\Gamma,0,0} = \alpha_{d,\mathbf{k},\Gamma} = M\prod_{j=1}^{d} \frac{1}{1+k_j^2}.$$
   As in Remark 4, this latter problem is quasi-polynomially tractable.. Hence $\mathrm{App}_{\Gamma,0,0}$ is at least quasi-polynomially tractable, as claimed. □

The right-hand inequality in Theorem 3 may be summarized as saying that our approximation problem $\mathrm{App}_{d,\Gamma,r,s}$ is no harder than the approximation problem $\mathrm{App}_{d,\Gamma,0,0}$ studied in [11]. The left-hand inequality tells us that $\mathrm{App}_{d,\Gamma,r,s}$ may be easier than $\mathrm{App}_{d,\Gamma,0,0}$. Despite this gap, we find that these two problems sometimes share the same level of tractability, as we shall see in what follows.

4.2 The Unweighted Case

If we specify the structure of the weights, we can get more detailed results. We first look at the unweighted case $\Gamma = \Gamma(\text{UNW})$, see item 1 in Remark 1. Our main result is that this problem is quasi-polynomially tractable.

Theorem 5. *Suppose that* $\Gamma = \Gamma(\text{UNW})$. *Let*

$$\tau^* = \frac{1}{\ln 2} \doteq 1.44270 \tag{37}$$

and

$$c_1 = \left(\sum_{j=-\infty}^{\infty} \left(\frac{1}{1+k^2} \right)^{\tau^*} \right)^{1/\tau^*} \doteq 2.09722.$$

Then

$$n(\varepsilon, \mathrm{App}_{d,\Gamma(\text{UNW}),r,s}) \le c_1 \exp\left(2\tau^*(1+\ln d)(1+\ln \varepsilon^{-1})\right),$$

and so $\mathrm{App}_{\Gamma(\text{UNW}),r,s}$ *is quasi-polynomially tractable. Moreover, its exponent is*

$$t^{\text{qpoly}}(\mathrm{App}_{\Gamma(\text{UNW}),r,s}) = 2\tau^* = \frac{2}{\ln 2} \doteq 2.88539.$$

Proof. From [7, Theorem 23.2], we have

$$t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma(\mathrm{UNW}),r,s}) = 2\inf\{\,\tau > 0 : C_\tau < \infty\},$$

where

$$C_\tau = \sup_{d\in\mathbb{N}} C_{\tau,d},$$

with

$$C_{\tau,d} = \frac{1}{d^2}\left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{d,\mathbf{k},\Gamma_1,r,s}^{\tau(1+\ln d)}\right)^{1/\tau}.$$

Moreover,

$$n(\varepsilon, \mathrm{App}_{\Gamma(\mathrm{UNW}),r,s}) \le C_\tau^\tau \exp\bigl(2\tau(1+\ln\varepsilon^{-1})(1+\ln d)\bigr)$$

for any $\tau > 0$ such that $C_\tau < \infty$. It suffices to show that τ^* is the minimal τ for which $C_\tau < \infty$.

Choose $\tau > 0$ such that $C_\tau < \infty$; we must show that $\tau \ge \tau^*$. For any $p > 0$, Theorem 2 tells us that

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{d,\mathbf{k},\Gamma(\mathrm{UNW}),r,s}^p \ge \left(\frac{1}{(s-r)!}\right)^p \sum_{\mathbf{k}\in\{0,1\}^d} \left(\frac{\alpha_{d,\mathbf{k},\Gamma(\mathrm{UNW})}}{\eta_{d,\mathbf{k}}^{s-r}}\right)^p.$$

But for $\mathbf{k}\in\{0,1\}^d$, we have

$$\eta_{d,\mathbf{k}} = 1 + \sum_{j=1}^d k_j^2 \le 1+d$$

and

$$\alpha_{d,\mathbf{k},\Gamma(\mathrm{UNW})} = \prod_{j=1}^d \frac{1}{1+k_j^2} = \left(\tfrac{1}{2}\right)^{|\{\,j\in[d]:k_j=1\,\}|}.$$

Hence for any $p \ge 0$, we have

$$\begin{aligned}\sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{d,\mathbf{k},\Gamma(\mathrm{UNW}),r,s}^p &\ge \left(\frac{1}{(s-r)!(1+d)^{s-r}}\right)^p \sum_{\mathbf{k}\in\{0,1\}^d} \left(\tfrac{1}{2}\right)^{p|\{\,j\in[d]:k_j=1\,\}|}\\ &= \left(\frac{1}{(s-r)!(1+d)^{s-r}}\right)^p \sum_{j=0}^d \binom{d}{j} \left(\tfrac{1}{2}\right)^{pj}\\ &= \left(\frac{1}{(s-r)!(1+d)^{s-r}}\right)^p \left[1+\left(\tfrac{1}{2}\right)^p\right]^d.\end{aligned}$$

Let $p = \tau(1+\ln d)$ and take logarithms. Then

$$\ln\left[\sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{d,\mathbf{k},\Gamma(\mathrm{UNW}),r,s}^{\tau(1+\ln d)}\right] \ge d\ln\left[1+\left(\tfrac{1}{2}\right)^{\tau(1+\ln d)}\right] - \tau(1+\ln d)\ln\bigl((s-r)!(1+d)^{s-r}\bigr).$$

Since $\ln(1+\delta) \ge \delta - \frac{1}{2}\delta^2$ for $\delta \ge 0$, we have

$$\ln\left[1+\left(\tfrac{1}{2}\right)^{\tau(1+\ln d)}\right] \ge \left(\tfrac{1}{2}\right)^{\tau(1+\ln d)}\left(1-\left(\tfrac{1}{2}\right)^{\tau(1+\ln d)+1}\right).$$

Since $d \in \mathbb{N}$ and $\tau > 0$, we have $\left(\frac{1}{2}\right)^{\tau(1+\ln d)+1} \le \left(\frac{1}{2}\right)^{\tau+1} \le \frac{1}{2}$, and so

$$\ln\left[1+\left(\tfrac{1}{2}\right)^{\tau(1+\ln d)}\right] \ge \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{\tau(1+\ln d)} = 2^{-(\tau+1)}d^{-\tau\ln 2}.$$

Without loss of generality, let $d \ge 2$, so that

$$\begin{aligned}\tau(1+\ln d)\ln\big((s-r)!(1+d)^{s-r}\big)&\\ &\le \tau\left(1+\frac{1}{\ln 2}\right)^2\ln^2 d + \tau\left(1+\frac{1}{\ln 2}\right)[\ln(s-r)! + (s-r)\ln d].\end{aligned}$$

Thus

$$\begin{aligned}\ln\Bigg[\sum_{\mathbf{k}\in\mathbb{Z}^d}\lambda_{d,\mathbf{k},\Gamma(\mathrm{UNW}),r,s}^{\tau(1+\ln d)}\Bigg] \ge 2^{-(\tau+1)}d^{1-\tau\ln 2} - \tau\left(1+\frac{1}{\ln 2}\right)^2\ln^2 d -&\\ \tau\left(1+\frac{1}{\ln 2}\right)[\ln(s-r)! + (s-r)\ln d]&,\end{aligned}$$

and so

$$\begin{aligned}\ln C_{\tau,d} &= \tau^{-1}\ln\Bigg[\sum_{\mathbf{k}\in\mathbb{Z}^d}\lambda_{d,\mathbf{k},\Gamma(\mathrm{UNW}),r,s}^{\tau(1+\ln d)}\Bigg] - 2\ln d\\ &\ge \tau^{-1}2^{-(\tau+1)}d^{1-\tau\ln 2}\\ &\quad - \left[\left(1+\frac{1}{\ln 2}\right)^2\ln^2 d + \left[3+\frac{1}{\ln 2}(s-r)\right]\ln d + \left(1+\frac{1}{\ln 2}\right)\ln(s-r)!\right].\end{aligned}$$

Since $\sup_{d\in\mathbb{N}} C_{\tau,d}$ must be finite, we see that the exponent of d must be non-positive. Hence we must have

$$\tau \ge \tau^* = \frac{1}{\ln 2} \doteq 1.44270,$$

as required.

It remains to show that $C_{\tau^*} < \infty$. From (27), it suffices to show that $C_{\tau^*} < \infty$ for $\mathrm{App}_{\Gamma,0,0}$. Suppose first that $d = 1$. Again using (27), we see that

$$\lambda_{1,k,\Gamma(\mathrm{UNW}),r,s} \le \lambda_{1,k,\Gamma(\mathrm{UNW}),0,0} = \frac{1}{1+k^2},$$

and so

$$C_{\tau^*,1}^{\tau^*} \le c_1 := \sum_{k\in\mathbb{Z}}\lambda_{1,k,\Gamma(\mathrm{UNW}),0,0}^{\tau^*} = \sum_{k=-\infty}^{\infty}\left(\frac{1}{1+k^2}\right)^{\tau^*}.$$

Since the terms in the series are $\Theta(k^{-2\tau^*})$, with $\tau^* \doteq 1.44270$, the series converges; using Mathematica, we find that $c_1 \doteq 2.09722$.

Now suppose that $d \geq 2$. Since

$$\lambda_{d,\mathbf{k},\Gamma(\text{UNW}),r,s} \leq \lambda_{d,\mathbf{k},\Gamma(\text{UNW}),0,0} \leq \alpha_{d,\mathbf{k},\Gamma(\text{UNW})} = \prod_{j=1}^{d} \frac{1}{1+k_j^2},$$

we have

$$\begin{aligned} C_{\tau^*,d}^{\tau^*} &\leq \frac{1}{d^2} \sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{d,\mathbf{k},\Gamma(\text{UNW}),0,0}^{\tau^*(1+\ln d)} \leq \frac{1}{d^2} \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} \cdots \sum_{k_d\in\mathbb{Z}} \left(\prod_{j=1}^{d} \frac{1}{1+k_j^2} \right)^{\tau^*(1+\ln d)} \\ &= \frac{1}{d^2} \left[\sum_{k=-\infty}^{\infty} \left(\frac{1}{1+k^2} \right)^{\tau^*(1+\ln d)} \right]^d . \end{aligned} \tag{38}$$

Since $d \geq 2$ and $\tau^* = 1/\ln 2$, we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left(\frac{1}{1+k^2} \right)^{\tau^*(1+\ln d)} &= \sum_{k=-\infty}^{\infty} (d\mathrm{e})^{-\ln(1+k^2)/\ln 2} \\ &= \frac{1}{d\mathrm{e}} \sum_{k=-\infty}^{\infty} (d\mathrm{e})^{-\ln[(1+k^2)/2]/\ln 2} \\ &\leq \frac{1}{d\mathrm{e}} \sum_{k=-\infty}^{\infty} (2\mathrm{e})^{-\ln[(1+k^2)/2]/\ln 2} \\ &= \frac{1}{d\mathrm{e}} \sum_{k=-\infty}^{\infty} \left(\frac{2}{1+k^2} \right)^{1+1/\ln 2} . \end{aligned} \tag{39}$$

Since the terms in the series

$$c_2 := \sum_{k=-\infty}^{\infty} \left(\frac{2}{1+k^2} \right)^{1+1/\ln 2} \tag{40}$$

are $\Theta(j^{-2(1+1/\ln 2)})$ and $2(1+1/\ln 2) > 1$, the series converges; again using Mathematica, we find that $c_2 \doteq 7.70707$. Combining (38)–(40), we find that

$$\sup_{d\geq 2} C_{\tau^*,d}^{\tau^*} \leq \sup_{d\geq 2} \frac{1}{d^2} \left(\frac{c_2}{d\mathrm{e}} \right)^d = \frac{1}{4} \left(\frac{c_2}{2\mathrm{e}} \right)^2 =: c_3,$$

where $c_3 \doteq 0.502423$, which is finite, completing the proof for the case $d \geq 2$. Combining the results for $d = 1$ and $d \geq 2$, we see that

$$C_{\tau^*} = \sup_{d\geq 2} C_{\tau^*,d} = \max\{c_1, c_3\}^{1/\tau^*} \doteq 1.67089,$$

as needed to prove the theorem. □

Remark 7. Note that the exponent of quasi-polynomial tractability is $2/\ln 2$, independent of the values of r and s. $\square$

4.3 Product Weights

In this section, we look at *product weights* $\Gamma(\Pi)$, which are defined by (2), subject to the condition (3) on the weightlets. As was the case for the space studied in [11, 12], we find that $H^{0,1}_{\Gamma(\Pi)}(\mathbb{T}^d) = \left[H^{0,1}_{\Gamma(\Pi)}(\mathbb{T})\right]^{\otimes d}$ has a tensor product structure for product weights, with

$$\alpha_{d,\mathbf{k},\Gamma} = \prod_{j=1}^{d} \frac{\gamma_{d,j}}{\gamma_{d,j} + k_j^2} \qquad \forall \mathbf{k} \in \mathbb{Z}^d.$$

In what follows, we shall assume that the weightlets $\gamma_{d,j}$ are uniformly bounded, i.e., that there exists $M > 0$ such that

$$\gamma_{d,j} \le M \qquad \forall\, j \in [d], d \in \mathbb{N}. \tag{41}$$

Remark 8. What happens if (41) does not hold? If we allow weightlets that are not uniformly bounded, then $\mathrm{App}_{\Gamma,r,s}$ can suffer from the curse of dimensionality. One such instance is given by choosing $\gamma_{d,j} \equiv d$ for all $j \in [d]$ and $d \in \mathbb{N}$. For a given $d \in \mathbb{N}$, let

$$\varepsilon_d = \frac{1}{2\sqrt{(s-r)!(1+d)^{s-r}(1+d^{-1})^d}} \sim \frac{1}{2\sqrt{(s-r)!\,\varepsilon\, d^{s-r}}} \qquad \text{as } d \to \infty.$$

Following the approach in [12, Section 5.2], we can show that $\lambda_{d,\mathbf{k},\Gamma,r,s} > \varepsilon_d^2$ for any $\mathbf{k} \in \{0,1\}^d$. Since $|\{0,1\}^d| = 2^d$, it follows that $n(\varepsilon_d, \mathrm{App}_{\Gamma,r,s}) \ge 2^d$. $\square$

4.3.1 Quasi-Poynomial Tractability

We claim that our approximation problem is always quasi-polynomially tractable for bounded product weights. Indeed, let Π_M denote product weights for which $\gamma_{d,j} \equiv M$. Then $\mathrm{App}_{\Gamma(\Pi),r,s}$ is no harder than $\mathrm{App}_{\Gamma(\Pi_M),r,s}$, since

$$\alpha_{d,\mathbf{k},\Gamma(\Pi)} \le \alpha_{d,\mathbf{k},\Gamma(\Pi_M)} = \prod_{j=1}^{d} \frac{1}{1 + k_j^2/M}.$$

It is now easy to see that $\mathrm{App}_{\Gamma(\Pi_M),r,s}$ is quasi-polynomially tractable, whence $\mathrm{App}_{\Gamma(\Pi)}$ is also quasi-polynomially tractable. The exponents of quasi-polynomial tractability satisfy

$$t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma(\Pi),r,s}) \le t^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma(\Pi_M),r,s}) = \frac{2}{\ln(1+M^{-1})}.$$

Moreover, the bound in this inequality is sharp, being attained by choosing equal weightlets $\Pi = \Pi_M$. To see why this is so, simply reiterate the proof of Theorem 5, replacing k_j^2 by k_j^2/M and $\frac{1}{2}$ by $1+1/M$ and making sure to use the upper bound $\eta_{d,\mathbf{k}} \le 1+d^2/M$.

4.3.2 Polynomial and Strong Polynomial Tractability

From Theorem 4, we see that since our weights are bounded, our approximation problem $\mathrm{App}_{\Gamma,r,s}$ is (strongly) polynomially tractable iff the same is true for the approximation problem $\mathrm{App}_{\Gamma,0,0}$. We now look at (strong) polynomial tractability in more detail:

Theorem 6. *We have the following results for bounded product weights.*

1. $\mathrm{App}_{\Gamma,r,s}$ *is strongly polynomially tractable iff there exists* $\tau > \frac{1}{2}$ *such that* $A_\tau < \infty$, *where*

$$A_\tau = \sup_{d\in\mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j}^{\tau}.$$

   a. *The exponent of strong polynomial tractability satisfies the inequality*

$$p(\mathrm{App}_{\Gamma,r,s}) \in \left[\max\left\{1, \frac{1}{s-r+1}\, p(\mathrm{App}_{\Gamma,0,0})\right\}, p(\mathrm{App}_{\Gamma,0,0})\right].$$

   Hence when $p^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,0,0}) = 1$, *we have*

$$p^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,r,s}) = p^{\mathrm{qpoly}}(\mathrm{App}_{\Gamma,0,0}).$$

   b. *Let*

$$p(\mathrm{App}_{\Gamma,0,0}) = 2\tau^*,$$

   where

$$\tau^* = \inf\{\, \tau > \tfrac{1}{2} : A_\tau < \infty \,\} \ge \tfrac{1}{2}.$$

   Then for all $\tau > \tau^*$, *we have*

$$n(\varepsilon, \mathrm{App}_{d,\Gamma,r,s}) \le n(\varepsilon, \mathrm{App}_{d,\Gamma,0,0}) \le \varepsilon^{-2\tau} \exp\left(2\,\zeta(2\tau)\pi^{-2\tau} A_\tau\right),$$

   where

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s}$$

   denotes the Riemann zeta function.

2. $\mathrm{App}_{\Gamma,r,s}$ *is polynomially tractable iff there exists* $\tau > \frac{1}{2}$ *such that* $B_\tau < \infty$, *where*

$$B_\tau = \limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_{d,j}^{\tau}.$$

When this holds, then for any $q_\tau > B_\tau$ *there exists a positive* C_τ *such that*

$$n(\varepsilon, S_d) \le C_\tau\, d^{q_\tau}\, \varepsilon^{-2\tau} \qquad \forall \varepsilon \in (0,1),\ d \in \mathbb{N}.$$

3. *For product weights independent of d, i.e., such that* $\gamma_{d,j} \equiv \gamma_j$ *for all* $d \in \mathbb{N}$*, strong polynomial tractability and polynomial tractability for* $\mathrm{App}_{\Gamma,r,s}$ *are equivalent.*

Proof. Follow the proof of [12, Thm. 5.3]. Take account of the following changes:

1. The factor π^2 in [12, Thm. 5.3] does not appear.
2. The expression $(k_j - 1)^2$ in [12, Thm. 5.3] becomes k_j^2.
3. Sums are over $\mathbb{Z}^d$ or $\mathbb{Z}$, rather than over $\mathbb{N}_0^d$ or $\mathbb{N}_0$. □

4.4 Bounded Finite-Order and Finite-Diameter Weights

As seen in Remark 3, if we allow unbounded weights, then we can run into situations in which $\mathrm{App}_{\Gamma,r,s}$ is strongly polynomially tractable, but $\mathrm{App}_{\Gamma,0,0}$ suffers from the curse of dimensionality. So we're only interested in *bounded* finite-order and finite-diameter weights, so that there exists $M > 0$ such that

$$M := \sup_{d\in\mathbb{N}} \sup_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} < \infty.$$

Now Theorem 3 tells us that our problem $\mathrm{App}_{d,\Gamma,r,s}$ is no harder than the problem $\mathrm{App}_{d,\Gamma,0,0}$. So we may follow the approach in the proof of [12, Theorem 5.4], which relies on [11, Theorem 4.1], to see that for any $\tau > 0$, there exist $C_{\tau,\omega} > 0$ such that

$$n(\varepsilon, \mathrm{App}_{\Gamma,r,s}) \le C_{\tau,\omega} M^{\tau/2} d^{\omega} \varepsilon^{-\tau}. \qquad (42)$$

Thus $\mathrm{App}_{\Gamma,r,s}$ is always polynomially tractable for finite-order weights. Finally, since finite-diameter weights are a special case of finite-order weights of order 1, we may substitute $\omega = 1$ in (42) to get a polynomially-tractable upper bound for $\mathrm{App}_{\Gamma,r,s}$ with finite-diameter weights.

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