DISCRETE MATH[1] W3203 Quiz 2

open book

SOLUTIONS

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Your Name (2 pts for legibly PRINTING your name)

Problem	Points	Score
your name	2	
1	20	
2	20	
3	18	
4	20	
5	20	
Total	100	

SUGGESTION: Do the EASIEST problems first!

HINT: Some of the solution methods involve highschool math as well as new methods from this class.

[1] An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.

1a (10 pts). Prove the following inequality:

$$2\sqrt{n+1} - \frac{1}{\sqrt{n+1}} > 2\sqrt{n} \quad \text{for all } n \geq 0.$$

PROOF:

$$\left(2\sqrt{n+1} - \frac{1}{\sqrt{n+1}}\right)^2 = 4(n+1) - 4 + \frac{1}{n+1}$$

$$= 4n + \frac{1}{n+1} > 4n = \left(2\sqrt{n}\right)^2$$

1b (10 pts). Now prove the following:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} \quad \text{for all } n \geq 1.$$

PROOF by induction: Basis(n=1): $\frac{1}{\sqrt{1}} = 1 < 2\sqrt{1} = 2$

Ind Hyp: Assume true for n.

Ind Step: $\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n+1}}$

$$= \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$$

$$< 2\sqrt{n+1} \ \ by\ part\ 1a.$$

2 (20 pts). Prove the following:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \quad \text{for } n \geq 0.$$

PROOF 1:

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=0}^{n} k\,\frac{n^{\underline{k}}}{k!} = n\sum_{k=0}^{n} \frac{(n-1)^{\underline{k-1}}}{\big(k-1\big)!} = n\sum_{k=0}^{n}\binom{n-1}{k-1} = n2^{n-1}$$

PROOF 2: Basis (n=0): Both sides are zero.

Ind Hyp: Assume true for n.

Ind Step:

$$\sum_{k=0}^{n+1} k\binom{n+1}{k} = \sum_{k=0}^{n+1} k\left[\binom{n}{k-1}+\binom{n}{k}\right] =$$

$$= \sum_{k=0}^{n+1} k\binom{n}{k-1} + \sum_{k=0}^{n+1} k\binom{n}{k} \; = \; \sum_{k=1}^{n+1} k\binom{n}{k-1} + \sum_{k=0}^{n} k\binom{n}{k}$$

$$= \left[\sum_{k=1}^{n+1}(k-1)\binom{n}{k-1} + \sum_{k=1}^{n+1}\binom{n}{k-1}\right] + \sum_{k=0}^{n} k\binom{n}{k}$$

$$= \left[\sum_{j=0}^{n} j\binom{n}{j} + \sum_{j=0}^{n}\binom{n}{j}\right] + \sum_{k=0}^{n} k\binom{n}{k}$$

$$= \; n2^{n-1} + 2^n + n2^{n-1} \; = \; n2^n + 2^n \quad \textit{by Ind Hyp}$$

$$= \; (n+1)2^n$$

3 (18 pts) We define the set S of binary strings recursively:

B. $1 \in S$

R. if $s \in S$, then $s01, s10$ and s^{-1}(reverse of s) $\in S$

3a (5). Show step-by-step how to construct a string in S with three consecutive 1's.

SOLUTION: $1 \rightarrow 110 \rightarrow 011 \rightarrow 01110$

3b (13). Prove that no string in S begins or ends in 111.

BASIS: 1 does not begin or end in 111.
IND HYP: Assume that no string derived in ≤n steps begins or ends in 111.
IND STEP: Consider a string $s \in S$, that begins or ends in 111, and whose derivation takes n+1 steps.
If the last step of that derivation was reversal, then s^{-1} was derived in only n steps, and cannot begin or end in 111, by the Ind Hyp. Thus, s cannot begin or end in 111.
Otherwise, either s = t01, or s = t10, where $t \in S$.
Thus, s ends either in x01 or in x10, not in 111.
Since t was derived in at most n steps, it cannot start in 111.

4 (20 pts). Consider the following recursion.

$$b_0 = 1$$
$$b_{n+1} = \binom{n}{0} b_n + \binom{n}{1} b_{n-1} + \binom{n}{2} b_{n-2} + \cdots + \binom{n}{n} b_0$$

4a (10). Calculate b_4.

SOLUTION:

$$b_1 = \binom{0}{0} b_0 = 1 \cdot 1 = 1 \qquad b_2 = \binom{1}{0} b_1 + \binom{1}{1} b_0 = 1 \cdot 1 + 1 \cdot 1 = 2$$
$$b_3 = \binom{2}{0} b_2 + \binom{2}{1} b_1 + \binom{2}{2} b_0 = 1 \cdot 2 + 2 \cdot 1 + 1 \cdot 1 = 5$$
$$b_4 = \binom{3}{0} b_3 + \binom{3}{1} b_2 + \binom{3}{2} b_1 + \binom{3}{3} b_0 = 1 \cdot 5 + 3 \cdot 2 + 3 \cdot 1 + 1 \cdot 1 = 15$$

4b (10). Prove that $b_{n+1} \geq 2^n$.

SOLUTION: Basis (n=0) $b_1 = 1 = 2^0$.
Ind Hyp: Assume true for all $j \leq n$.
Ind Step: Then

$$b_{n+1} = \binom{n}{0} b_n + \binom{n}{1} b_{n-1} + \binom{n}{2} b_{n-2} + \cdots + \binom{n}{n} b_0$$
$$\geq \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

5 (20 pts). A bakery sells four varieties of bagels: egg, onion, poppy, and sesame.

5a (10). How many different combinations of six bagels are possible?

SOLUTION: $\binom{4+6-1}{6} = \binom{9}{3} = 84$

5b (10). How many of the combinations in 5a contain at least four bagels of one kind?

SOLUTION: $4\binom{4+2-1}{2} = 4\binom{5}{2} = 40$