**DISCRETE MATH W3203 Quiz 2**  
open book

**SOLUTIONS**

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**Your Name (2 pts for LEGIBLY PRINTING your name on this line)**

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**Total** 100

**SUGGESTION:** Do the EASIEST problems first!

**HINT:** Some of the solution methods involve highschool math as well as new methods from this class.

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\*An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.*
1 (13 pts). You have infinitely many 3.5¢ stamps and 6.5¢ stamps. Prove that you can make any integer postage of at least 35¢.

PROOF: by induction
BASIS: \(35 = 10 \times 3.5\)

IND HYP: Assume that \(3.5r + 6.5s = n \geq 35\).

IND STEP: two cases

Case 1 \(s \geq 2\) : Then replace two 6.5¢ stamps by four 3.5¢ stamps.
\[
3.5(r + 4) + 6.5(s - 2) = 3.5r + 14 + 6.5s - 13
\]
\[
= 3.5r + 6.5s + 1 = n + 1
\]

Case 2 \(s < 2\). Then \(3.5r > 35 - 6.5 = 28.5\).
\[
\therefore r > \frac{28.5}{3.5} \quad \therefore r \geq \left\lfloor \frac{28.5}{3.5} \right\rfloor = 9.
\]
Then replace 9 3.5¢ stamps by 5 6.5¢ stamps.
\[
3.5(r - 9) + 6.5(s + 5) = 3.5r - 31.5 + 6.5s + 32.5
\]
\[
= 3.5r + 6.5s + 1 = n + 1.
\]
2 (20 pts). Let \( T \) be the set of all bitstrings with the same number of 0’s as 1’s. We recursively define the set \( S \) of bitstrings:

B. \( \lambda \in S \)

R. If \( x, y \in S \), then

1. \( 0x1 \in S \); 2. \( 1x0 \in S \); 3. \( xy \in S \).

Prove that \( T \subseteq S \).

PROOF. Use induction on the length of string \( z \in T \).

Basis. \( \text{len}(z) = 0 \Rightarrow z = \lambda \). By (B), \( z \in S \).

Ind Hyp. Suppose \((\forall z \in T)[\text{len}(z) \leq n \Rightarrow z \in S] \).

Ind Step. Let \( z \in T \) with \( \text{len}(z) = n + 2 \).

Split into two cases.

Case 1. \( z = 0w1 \) or \( z = 1w0 \), and \( \text{len}(w) = n \). \( \therefore w \in T \). Then \( w \in S \), by IndHyp.

Hence, by Rules (1) and (2), \( 0w1, 1w0 \in S \).

Case 2a. \( z = 0a_1 \cdots a_n 0 \).

Let \( r \) be the smallest subscript s.t. the prefix \( x = 0a_1 \cdots a_r \) has as many 1’s as 0’s. It follows that the suffix \( y = a_{r+1} \cdots a_n 0 \) also has as many 1’s as 0’s. By Ind Hyp, \( x, y \in S \).

By Rule (3), \( z = xy \in S \).

Case 2b. \( z = 1a_1 \cdots a_n 1 \). Similar proof to Case 2a.
3 (20). Let $S$ be the set of 6-digit decimal strings 000000 to 999999.

3a (10) How many strings in $S$ use only two different digits?

SOLUTION:

$$\binom{6}{1}10^2 + \binom{6}{2}10^2 + \frac{1}{2}\binom{6}{3}10^2 = [6 + 15 + 10] \times 90 = 31 \times 90$$

3b (10) How many strings in $S$ use three, four, or five different digits?

SOLUTION:

$$10^6 - \binom{6}{1}10^2 \binom{6}{2}10^2 \binom{6}{3}10^2 \left[ \binom{6}{1} + \binom{6}{2} + \frac{1}{2}\binom{6}{3} \right] - \frac{10}{6}$$
4 (10pts). M&M candies come in red, blue, yellow, and green. In how many ways can you select a bag of 20 M&M’s? (Order of selection does not matter.)

SOLUTION: \[ \binom{23}{3} \]

\[ \underbrace{0 \cdots 010 \cdots 010 \cdots 0}_{\text{23 bits with 3 1's}} \]

red blue yellow green
5a (15 pts) Jessica tosses a fair coin three times. Joshua also tosses a fair coin three times. What is the probability that they both tossed the same number of heads?

SOLUTION:

\[
\frac{1 \cdot 1 + 3 \cdot 3 + 3 \cdot 3 + 1 \cdot 1}{8 \cdot 8} = \frac{20}{64} = \frac{5}{16}
\]

5b (5 pts) Given that they both toss the same number of heads, what is the probability that it is exactly one head?

SOLUTION:

\[
p(1H \mid \text{same #H}) = \frac{p(1H \wedge \text{same #H})}{p(\text{same #H})} = \frac{3 \cdot 3/64}{20/64} = \frac{9}{20}
\]
6 (15 pts). A tetrahedral (i.e., 4-sided) die is marked with 1, 2, 3, and 4 spots on its four sides. It is loaded so that $pr(j) = (5-j)/10$.

6a (5 pts). Calculate the expected value of a roll of this die.

**SOLUTION**: 

\[
E(X) = \frac{1}{10} \cdot 4 + \frac{2}{10} \cdot 3 + \frac{3}{10} \cdot 2 + \frac{4}{10} \cdot 1 = \frac{20}{10} = 2
\]

6b (10 pts). Calculate the standard deviation.

**SOLUTION**: 

\[
V(X) = (1-2)^2 \cdot \frac{4}{10} + (2-2)^2 \cdot \frac{3}{10} + (3-2)^2 \cdot \frac{2}{10} + (4-2)^2 \cdot \frac{1}{10}
\]

\[
= 1 \cdot \frac{4}{10} + 0 \cdot \frac{3}{10} + 1 \cdot \frac{2}{10} + 4 \cdot \frac{1}{10} = \frac{10}{10} = 1
\]

\[
\sigma(X) = \sqrt{V(X)} = \sqrt{1} = 1
\]