DISCRETE MATH[1] W3203 Quiz 2

open book

SOLUTIONS

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Your Name (2 pts for LEGIBLY PRINTING your name on this line)

Problem	Points	Score
your name	2	
1	13	
2	20	
3	20	
4	10	
5	20	
6	15	
Total	100	

SUGGESTION: Do the EASIEST problems first!

HINT: Some of the solution methods involve highschool math as well as new methods from this class.

[1] An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.

1 (13 pts). You have infinitely many 3.5¢ stamps and 6.5¢ stamps. Prove that you can make any integer postage of at least 35¢.

PROOF: by induction
BASIS: $35 = 10 \times 3.5$

IND HYP: Assume that $3.5r + 6.5s = n \geq 35$.

IND STEP: two cases

Case 1 $s \geq 2$: Then replace two 6.5¢ stamps by four 3.5¢ stamps.

$$3.5(r+4) + 6.5(s-2) \;=\; 3.5r + 14 + 6.5s - 13$$

$$=\; 3.5r + 6.5s + 1 \;=\; n+1$$

Case 2 $s < 2$. Then $3.5r > 35 - 6.5 = 28.5$.

$$\therefore r > \frac{28.5}{3.5} \quad \therefore r \geq \left| \frac{28.5}{3.5} \right| = 9. \quad \text{Then}$$

replace 9 $3.5¢$ – stamps by 5 $6.5¢$ stamps.

$$3.5(r-9) + 6.5(s+5) \;=\; 3.5r - 31.5 + 6.5s + 32.5$$

$$=\; 3.5r + 6.5s + 1 \;=\; n+1.$$

2 (20 pts). Let T be the set of all bitstrings with the same number of 0's as 1's. We recursively define the set S of bitstrings:

B. $\lambda \in S$

R. If $x, y \in S$, then

(1) $0x1 \in S$; (2) $1x0 \in S$; (3) $xy \in S$.

Prove that $T \subseteq S$.

PROOF. Use induction on the length of string $z \in T$.

Basis. $\text{len}(z) = 0 \Rightarrow z = \lambda$. By (B), $z \in S$.

Ind Hyp. Suppose $(\forall z \in T)[\text{len}(z) \le n \Rightarrow z \in S]$.

Ind Step. Let $z \in T$ with $\text{len}(z) = n + 2$.

Split into two cases.

Case 1. z = 0w1 or z = 1w0, and len(w) = n. $\therefore$ $w \in T$.
Then $w \in S$, by IndHyp.

Hence, by Rules (1) and (2), $0w1,\ 1w0 \in S$.

Case 2a. $z = 0a_1 \cdots a_n 0$.

Let r be the smallest subscript s.t. the prefix $x = 0a_1 \cdots a_r$ has as many 1's as 0's. It follows that the suffix $y = a_{r+1} \cdots a_n 0$ also has as many 1's as 0's. By Ind Hyp, $x, y \in S$.

By Rule (3), $z = xy \in S$.

Case 2b. $z = 1a_1 \cdots a_n 1$. Similar proof to Case 2a.

3 (20). Let S be the set of 6-digit decimal strings 000000 to 999999.

3a (10) How many strings in S use only two different digits?

SOLUTION:

$$\binom{6}{1}10^{\underline{2}}+\binom{6}{2}10^{\underline{2}}+\frac{1}{2}\binom{6}{3}10^{\underline{2}}=[6+15+10]\times 90=31\times 90$$

3b (10) How many strings in S use three, four, or five different digits?

SOLUTION:

$$10^6-\underbrace{10^{\underline{6}}}_{\text{all six different}}-\underbrace{\left\lfloor\binom{6}{1}10^{\underline{2}}+\binom{6}{2}10^{\underline{2}}+\binom{6}{3}10^{\underline{2}}\right\rfloor}_{\text{two different digits}}-\underbrace{10}_{\text{all same}}$$

4 (10pts). M&M candies come in red, blue, yellow, and green. In how many ways can you select a bag of 20 M&M's? (Order of selection does not matter.)

SOLUTION: $\binom{23}{3}$

$$\underbrace{0\cdots0}_{\text{red}}1\underbrace{0\cdots0}_{\text{blue}}1\underbrace{0\cdots0}_{\text{yellow}}1\underbrace{0\cdots0}_{\text{green}} \quad 23 \text{ bits with } 3 \text{ 1's}$$

5a (15 pts) Jessica tosses a fair coin three times. Joshua also tosses a fair coin three times. What is the probability that they both tossed the same number of heads?

SOLUTION:

$$\frac{1\cdot 1+3\cdot 3+3\cdot 3+1\cdot 1}{8\cdot 8}=\frac{20}{64}=\frac{5}{16}$$

5b (5 pts) Given that they both toss the same number of heads, what is the probability that it is exactly one head?

SOLUTION:

$$p(1\mathrm{H} \mid \text{same } \#\mathrm{H})=\frac{p(1\mathrm{H}\wedge \text{same } \#\mathrm{H})}{p(\text{same } \#\mathrm{H})}=\frac{3\cdot 3/64}{20/64}=\frac{9}{20}$$

6 (15 pts). A tetrahedral (i.e., 4-sided) die is marked with 1, 2, 3, and 4 spots on its four sides. It is loaded so that pr(j) = (5-j)/10.

6a (5 pts). Calculate the expected value of a roll of this die.

SOLUTION:

$$E(X) = 1 \cdot \frac{4}{10} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{2}{10} + 4 \cdot \frac{1}{10} = \frac{20}{10} = 2$$

6b (10 pts). Calculate the standard deviation.

SOLUTION:

$$V(X) = (1-2)^2 \cdot \frac{4}{10} + (2-2)^2 \cdot \frac{3}{10} + (3-2)^2 \cdot \frac{2}{10} + (4-2)^2 \cdot \frac{1}{10}$$

$$= 1 \cdot \frac{4}{10} + 0 \cdot \frac{3}{10} + 1 \cdot \frac{2}{10} + 4 \cdot \frac{1}{10} = \frac{10}{10} = 1$$

$$\sigma(X) = \sqrt{V(X)} = \sqrt{1} = 1$$