Professor J. L. Gross
CS W3203 Discrete Math

DISCRETE MATH[1] W3203 Quiz 1

open book

SOLUTIONS

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Your Name (2 pts for LEGIBLY PRINTING your name on this line)

Problem	Points	Score
your name	2	
1	10	
2	15	
3	18	
4	15	
5	20	
6	20	
Total	100	

SUGGESTION: Do the EASIEST problems first!

HINT: Some of the solution methods involve highschool math as well as new methods from this class.

[1] An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.

1 (13 pts). Use a truth table to decide whether this argument is valid.

$$\begin{array}{l} q \to p \\ \neg q \to r \\ \underline{\neg p} \\ \therefore \; r \end{array}$$

SOLUTION:

p	q	r	$\neg p$	$q \to p$	$\neg q$	$\neg q \to r$		r
T	T	T	F					
T	T	F	F					
T	F	T	F					
T	F	F	F					
F	T	T	T	F				
F	T	F	T	F				
F	F	T	T	T	T	T	*	T
F	F	F	T	T	T	F		

2a (15 pts). Decide which of the following functions are one-to-one.

2a (5). $f(x) = 3x \mod 7$ from the set $\{0,1,\ldots,6\}$ to itself.
SOLUTION: yes

x	0	1	2	3	4	5	6
f(x)	0	3	6	2	5	1	4

2b (5). $f(x) = 4x \mod 6$ from the set $\{0,1,\ldots,5\}$ to itself.
SOLUTION: no

x	0	1	2	3	4	5
f(x)	0	4	2	0	4	2

2c (5). $f(x) = \lfloor \pi x \rfloor \mod 7$ from the set $\{0,1,\ldots,6\}$ to itself.
SOLUTION: yes

x	0	1	2	3	4	5	6
f(x)	0	3	6	2	5	1	4

3 (18 pts). Prove that $\left|\left(\frac{-1}{2}\right)^{\underline{n}}\right|$ asymptotically dominates 2^n.

3a (6). First express $\left|\left(\frac{-1}{2}\right)^{\underline{n}}\right|$ as a product of n factors.

SOL: $\left|\left(\frac{-1}{2}\right)^{\underline{n}}\right| = \frac{1}{2} \bullet \frac{3}{2} \bullet \cdots \bullet \frac{2n-1}{2} = \prod_{j=1}^{n} \left(\frac{2j-1}{2}\right)$

3b (6). Next find a value K for which $\left|\left(\frac{-1}{2}\right)^{\underline{K}}\right| > 2^K$.

SOLUTION: K = 6 is the smallest possible.

$$\left|\left(\frac{-1}{2}\right)^{\underline{6}}\right| = \frac{1 \bullet 3 \bullet 5 \bullet 7 \bullet 9 \bullet 11}{2^6} > \frac{1 \bullet 2 \bullet 2^2 \bullet 7 \bullet 2^3 \bullet 11}{2^6} = 77 > 2^6$$

3c (6). Prove that $\left|\left(\frac{-1}{2}\right)^{\underline{K+r}}\right| > 2^{K+r}$ for all $r > 0$.

Proof: $$\left|\left(\frac{-1}{2}\right)^{\underline{K+r}}\right| = \left|\left(\frac{-1}{2}\right)^{\underline{K}}\right| \bullet \prod_{j=K+1}^{K+r} \overbrace{\left(\frac{2j-1}{2}\right)}^{>2 \text{ for } j>6} > 2^K \bullet 2^r = 2^{K+r}$$

4a (10pts). Convert 1246_7 to a base-6 numeral.
ZERO CREDIT unless your work is shown.

SOL: $1246_7 = 6 \cdot 1 + 4 \cdot 7 + 2 \cdot 49 + 1 \cdot 343$

$$= 6 + 28 + 98 + 343 = 475_{10}$$

$6\overline{)475}$ = 79 R1 $\quad 6\overline{)79}$ = 13 R1 $\quad 6\overline{)13}$ = 2 R1 $\quad 6\overline{)2}$ = 0 R2 $\quad \Rightarrow \quad 2111_6$

Check: $1 + 6 + 36 + 432 = 475_{10}$

4b (5pts). Show that 667_{77} is divisible by 19.
ZERO CREDIT unless your work is shown.
Hint: First expand as a polynomial in 77, and then substitute 77= 4•19+1.

SOL: $667_{77} = 6 \cdot 77^2 + 6 \cdot 77 + 7$

$$= 6 \cdot (4 \cdot 19 + 1)^2 + 6 \cdot (4 \cdot 19 + 1) + 7$$

$$= 6 \cdot \left(4^2 19^2 + 2 \cdot 4 \cdot 19 + 1\right) + 6 \cdot (4 \cdot 19 + 1) + 7$$

$$= 6 + 6 + 7 + 19M \;\; = 19 + 19M \text{ divisible by } 19$$

5a (6). Calculate gcd(3122, 833) by prime power factorization.

SOL: $3122 = 2 \cdot 1561 = 2 \cdot 7 \cdot 223$

$833 = 7 \cdot 119 = 7^2 \cdot 17$

$\text{gcd}(3122, 833) = 7$

5b (8). Calculate gcd(3122, 833) by the Euclidean algorithm.

SOL:

$$\begin{array}{ccccccc} 3122 & = & 833 & \cdot & 3 & + & 623 \\ 833 & = & 623 & \cdot & 1 & + & 210 \\ 623 & = & 210 & \cdot & 2 & + & 203 \\ 210 & = & 203 & \cdot & 1 & + & 7 \\ 203 & = & 7 & \cdot & 29 & + & 0 \end{array}$$

$\text{gcd}(3122, 833) = 7$

5c (6). Find integers M and N such that
$3122M + 833N = \text{gcd}(3122, 833)$.

SOL: Find M, N s.t. $\frac{3122}{7}\text{M} + \frac{833}{7}\text{N} = \frac{\text{gcd}(3122,833)}{7} = 1$

i.e., $446\text{M} + 119\text{M} = 1$

$4 \cdot 119 = 476 \quad 476 - 446 = 30$

$1 = 4 \cdot 30 - 119 = 4(4 \cdot 119 - 446) - 119 = \underline{\underline{15}} \cdot 119 - \underline{\underline{4}} \cdot 446$

6a (10 pts). Calculate $147^{810} \bmod 13$.

$$\text{SOL}:\ 147^{810} \bmod 13 \equiv 147^{6} \bmod 13 \ (\text{since } 810 \equiv 6 \mod 12)$$

$$\equiv 4^{6} \bmod 13 \ (\text{since } 147 \equiv 4 \mod 13)$$

$$\equiv 3^{3} \bmod 13 \ \equiv 27 \bmod 13 \ \equiv 1 \bmod 13$$

6b (5 pts). Find the multiplicative inverse of 13 mod 17.

SOL:

x	1	2	3	4	$\cdots$
13x mod 17	13	9	5	1	$\cdots$

$$4 \cdot 13 = 52 \equiv 3 \cdot 17 + 1 \equiv 1 \mod 17$$

6c (5 pts). Find the multiplicative inverse of 41 mod 51.
Hint: Find M such that 41M + 51N = 1.

SOLUTION: 5•41 – 4•51 = 1
Thus, 5 is the inverse of 41 mod 51.