Professor J. L. Gross
CS W3203 Discrete Math

DISCRETE MATH[1] W3203 Final Exam

open book

SOLUTIONS

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Your Name (2 pts for LEGIBLY PRINTING your name on this line)

Problem	Points	Score
your name	2	
1	18	
2	20	
3	20	
4	25	
5	25	
6	30	
7	35	
8	25	
Total	200	

SUGGESTION: Do the EASIEST problems first!
HINT: Some of the solution methods involve highschool math as well as new methods from this class.

[1] An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.

1a (3 pts). For the following recursion, calculate a_3, a_4 and a_5.

$$a_0 = 0,\ a_1 = 1,\ a_2 = 2; \quad a_n = \frac{n}{n-1} a_{n-1} + \frac{n}{n-2} a_{n-2} \quad \text{for } n \ge 3$$

SOL. $a_3 = \frac{3}{2} \cdot 2 + \frac{2}{1} \cdot 1 = 6;$

$a_4 = \frac{4}{3} \cdot 6 + \frac{4}{2} \cdot 2 = 12;\ a_5 = \frac{5}{4} \cdot 12 + \frac{5}{3} \cdot 6 = 25;$

1b (15). Use induction to prove that $a_n = nf_n$, where f_n is the $n^{\underline{th}}$ number in the Fibonacci sequence 0, 1, 1, 2, 3, 5, … .

BASIS: true for a_0, a_1, and a_2

IND HYP: assume true for a_{n-1} and a_{n-2} $n \ge 3$

IND STEP: $a_n = \frac{n}{n-1} a_{n-1} + \frac{n}{n-2} a_{n-2}$

$$= \frac{n}{n-1}(n-1)f_{n-1} + \frac{n}{n-2}(n-2)f_{n-2}$$

$$= nf_{n-1} + nf_{n-2} = n(f_{n-1} + f_{n-2}) = nf_n$$

2 (20 pts). Solve the following recursion:

$$a_0 = 2, \ \ a_1 = 3; \qquad 6a_n = 5a_{n-1} - a_{n-2}$$

SOL. $6r^2 - 5r + 1 = (3r-1)(2r-1) = 0 \quad r = \frac{1}{3}, \ \frac{1}{2}$

$$a_n = B\left(\frac{1}{3}\right)^n + C\left(\frac{1}{2}\right)^n$$

$$\begin{aligned} a_0 &= 2 = B + C \\ a_1 &= 3 = \frac{B}{3} + \frac{C}{2} \end{aligned} \quad \text{solve} \quad \begin{aligned} B &= -12 \\ C &= 14 \end{aligned}$$

$$a_n = (-12)\left(\frac{1}{3}\right)^n + 14\left(\frac{1}{2}\right)^n$$

3 (20 pts). Calculate the value of the general coefficient a_n in the power series expansion $\dfrac{2-3x}{1-10x+21x^2} = \sum_{n=0}^{\infty} a_n x^n$

SOL. partial fractions

$$\frac{2-3x}{1-10x+21x^2} = \frac{2-3x}{(1-3x)(1-7x)} = \frac{A}{1-3x} + \frac{B}{1-7x}$$

$$\begin{aligned} A + B &= 2 \\ -7A - 3B &= -3 \end{aligned} \quad \text{solve} \quad \begin{aligned} A &= -\frac{3}{4} \\ B &= \frac{11}{4} \end{aligned}$$

$$a_n = -\frac{3}{4} \cdot 3^n + \frac{11}{4} \cdot 7^n$$

4 (25). For two positive integers, we write $m \prec n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance, $75 \prec 14$, because $3 + 5 \leq 2 \cdot 7$.

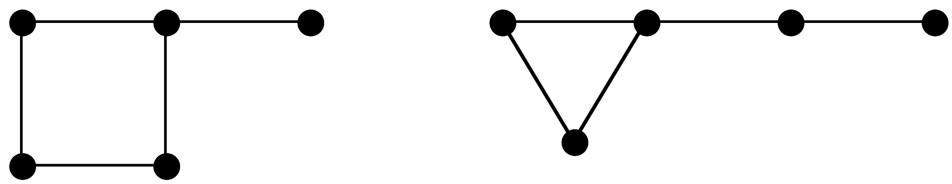
4a (5). Is this relation reflexive? Explain.
SOL. Yes, because the product of positive integers greater than or equal to 2 is less than their sum.

4b (10). Is this relation anti-symmetric? Explain.
SOL. No, because $33 \prec 26$ and $26 \prec 33$, but $26 \neq 33$.

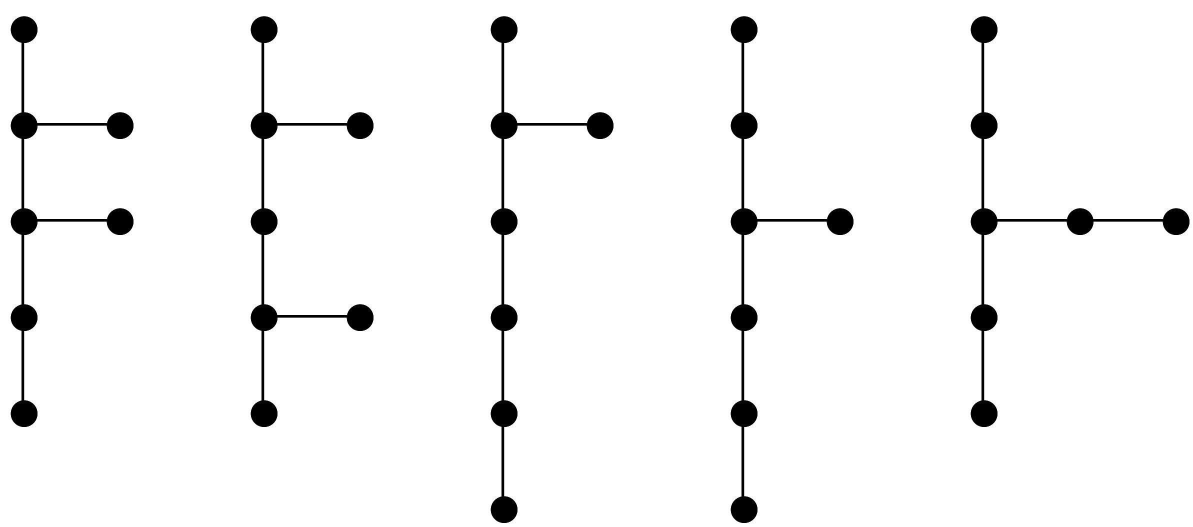
4c (10). Is this relation transitive? Explain.
SOL. No, because $33 \prec 35$ and $35 \prec 13$, but $\neg(33 \prec 13)$.

5a (10). Draw two non-isomorphic 5-vertex, 5-edge simple graphs with the same degree sequence.

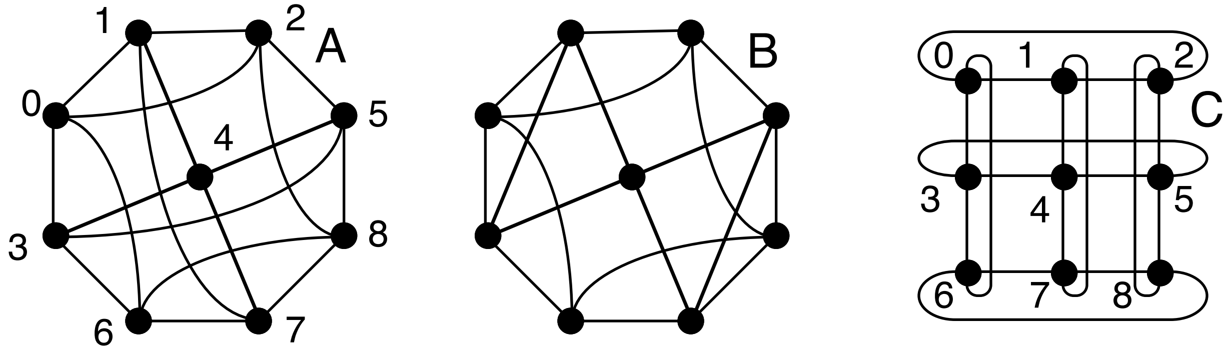
SOLUTION.



5b (15). Draw all possible 7-vertex trees with maximum degree 3.
SOLUTION. The degree seq is either 3321111 or 3222111.



6 (30pts). Which pairs of these graphs are isomorphic. Explain.

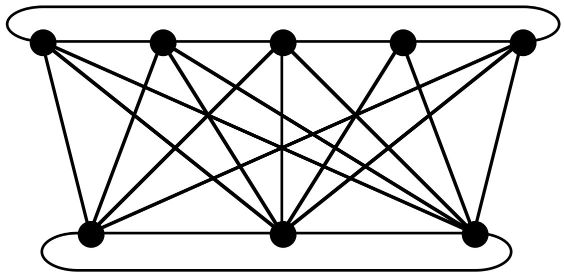


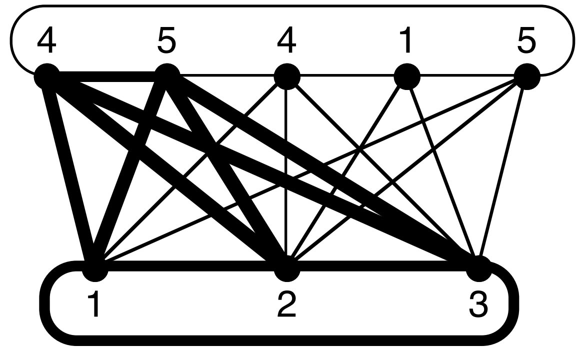
SOLUTION. A $\cong$ C as shown by vertex labelling above.
Zillions of reasons why A and B are non-isomorphic. E.g.,

(1) A is nonplanar (see problem #8) but B is planar.

(2) B has pairs of 3-cycles that share an edge, but C does not.

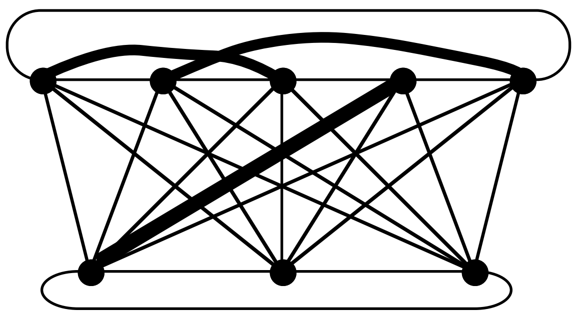
7a (10 pts). Calculate the chromatic number of this graph?





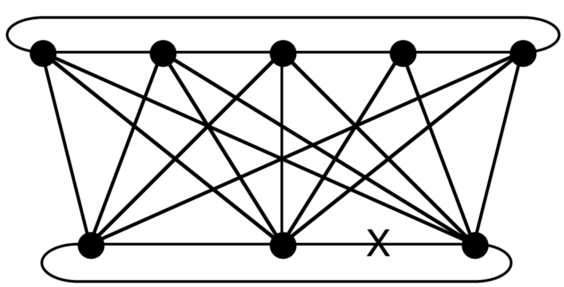
SOL. Given 5-coloring shows 5 is upper bound. Bold K_5 shows 5 is lower bound.

7b (15). Show three place where adding a single edge would increase the chromatic number.



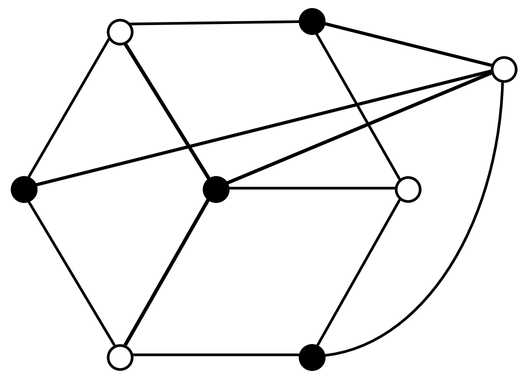
SOL. Of the six missing edges, these three work.

7c (10). Show where to delete an edge to decrease the chromatic number.



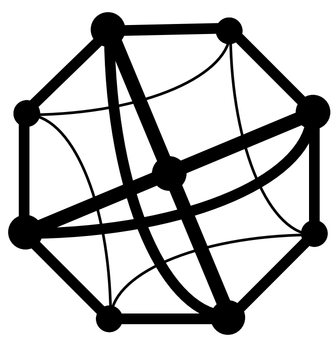
SOL. Delete the marked edge. Then start with 5-coloring of part (a), and recolor the bottom right vertex with color 2.

8a (10 pts). Decide whether the following graph is planar.



SOL. Easy proof – bipartite as shown. E = 13. V= 8.
13 = E > 2V- 4 = 12
Harder Proof. Draw $K_{3,3}$

8b (15 pts). Decide whether the following graph is planar.



Solution. NOT planar. But NOT bipartite. Must find a Kuratowski graph. K_5 is shown.