DISCRETE MATH[1] W3203 FINAL EXAM

open book

SOLUTIONS

____
Your Name (2 pts for legibly PRINTING your name)

Problem	Points	Score
your name	2	
1	23	
2	20	
3	25	
4	25	
5	30	
6	30	
7	20	
8	25	
Total	200	

SUGGESTION: Do the EASIEST problems first!

HINT: Some of the solution methods involve highschool math as well as new methods from this class.

[1] An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.

1 (23 pts). Consider the following recurrence system:

$$a_0 = 0; \quad a_1 = 1;$$
$$a_n = 4a_{n-2} + 1$$

1a (2). Calculate $a_2, \ a_3, \ a_4, \ and \ a_5$.

$$SOL: \quad a_2 = 4a_0 + 1 = 4 \cdot 0 + 1 = 1 \quad a_3 = 4a_1 + 1 = 4 \cdot 1 + 1 = 5$$
$$a_4 = 4a_2 + 1 = 4 \cdot 1 + 1 = 5 \quad a_5 = 4a_3 + 1 = 4 \cdot 5 + 1 = 21$$

1b (21). Find a closed form for a_n.

SOL 1 (by special forms):

Homogenous	*Particular*
$\hat{a}_n - 4\hat{a}_{n-2} = 0$	$\dot{a}_n - 4\dot{a}_{n-2} = 1$
$r^2 - 4 = 0$	$try \quad \dot{a}_n = D$
$\hat{a}_n = B \cdot 2^n + C$	$D - 4D = 1; \quad D = -\frac{1}{3}$

$$a_0 = 0 = B \cdot 2^0 + C(-2)^0 - \frac{1}{3} \qquad \left.\begin{aligned} B + C &= \frac{1}{3} \\ 2B - 2C &= \frac{4}{3} \end{aligned}\right\} \qquad B = \frac{1}{2}$$
$$a_1 = 1 = B \cdot 2^1 + C(-2)^1 - \frac{1}{3} \qquad\qquad\qquad C = -\frac{1}{6}$$

$$\underline{\underline{a_n = 2^{n-1} - \frac{1}{3} \cdot (-2)^n - \frac{1}{3}}}$$

SOL 2 by gen fcns on next page

SOL 2 by gen fcns:

$$\sum_{n=2}^{\infty} a_n x^n = 4\sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + x^2 \sum_{n=2}^{\infty} x^{n-2}$$

$$A(x) - a_1 x - a_0 = 4x^2 A(x) + \frac{x^2}{1-x}$$

$$A(x) - x = 4x^2 A(x) + \frac{x^2}{1-x}$$

$$A(x)\left[1 - 4x^2\right] = \frac{x^2}{1-x} + x = \frac{x}{1-x}$$

$$A(x) = \frac{x}{(1-2x)(1+2x)(1-x)} = \frac{B}{1-2x} + \frac{C}{1+2x} + \frac{D}{1-x}$$

$$solve \;\; with \;\; lin \;\; eq: B = \frac{1}{2},\; C = -\frac{1}{6},\; D = -\frac{1}{3}$$

$$\underline{\underline{a_n = 2^{n-1} - \tfrac{1}{3} \cdot (-2)^n - \tfrac{1}{3}}}$$

2 (20 pts). Consider this binary relation on $\{1, 2, 3, 4\}$:

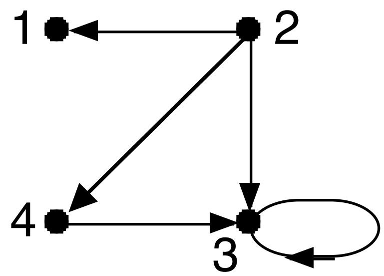
$$R = \{(2,1),(2,3),(2,4),(3,3),(4,3)\}$$

2a (5). Represent R as a matrix.

$$SOL: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2b (5). Represent R as a digraph.

SOL:



2c (10). Calculate the relation R^2.

$$SOL: \quad R^2 = \{ \ (2,3), \ (3,3), \ (4,3) \ \}$$

3a (10 pts). Give an example of a relation R on the set

$$S = \{a,\ b,\ c,\ d\}$$

such that R is both an equivalence relation and a partial ordering.

$SOL:\quad R = \{(a,a),\ (b,b),\ (c,c),\ (d,d)\}$

3b (15 pts). Prove that there is only one such relation R.

Proof: Let Q be any eq rel on S that is also a p.o.

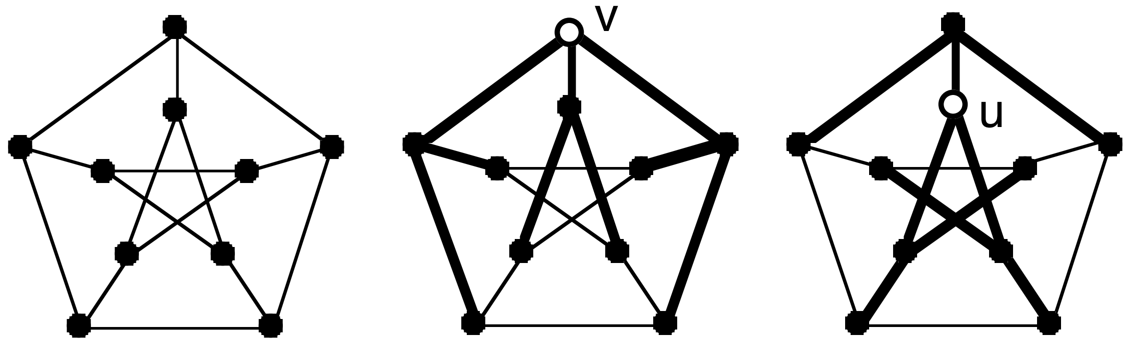
Q Reflexive $\Rightarrow\ (\forall x \in S)[(x, x) \in Q].\ \ \therefore R \subseteq Q.$

Now suppose that $(x, y) \in Q.\quad Symmetry \Rightarrow (y, x) \in Q.$

But then anti $-$ symmetry $\Rightarrow x = y.\ \ \therefore Q \subseteq R.$

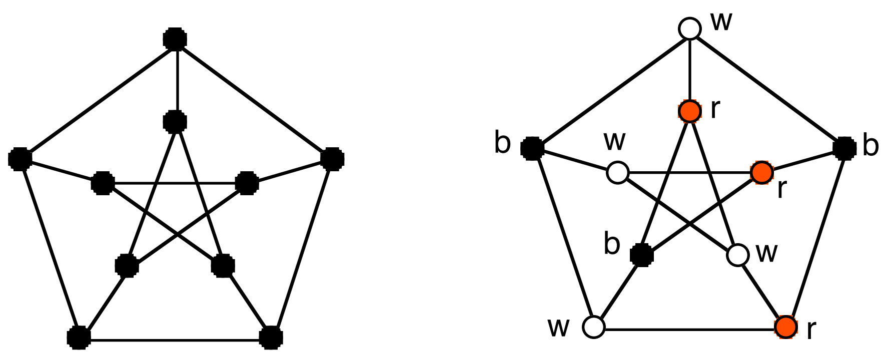
$\therefore R = Q.$

4a (10). The *diameter* of a graph is the maximum, taken over all vertex pairs u, v of the distance between u and v. Calculate the diameter of the Petersen graph.



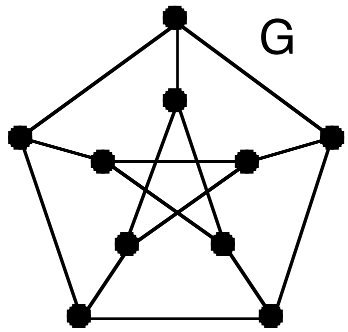
SOL: The two spanning trees both have height = 2.
Thus vertices u and v are within distance 2 of every other vertex.
By symmetry, no vertex has eccentricity ≥ 2.
Thus, the diameter = 2.

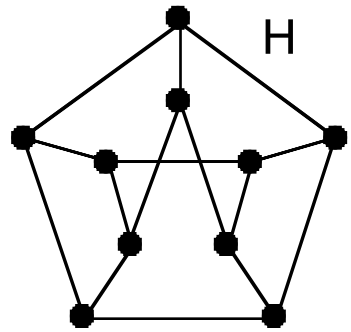
4b (15). Calculate the chromatic number of the Petersen graph.

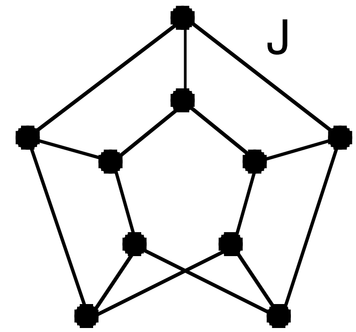


SOL: A 3-coloring is shown at the right.
Since there is a 5-cycle, no 2-coloring is possible.
Thus, the chromatic number is 3.

5 (30 pts). Consider the following three graphs:







5a (10). Decide whether G and H are isomorphic. Explain.

SOL (many ways): NO.
(1) diameter: diam (G) = 2; diam(J) = 3.
(2) girth: girth (G) = 5; H has 4-cycles.
(3) G is vertex-transitive; H is not.

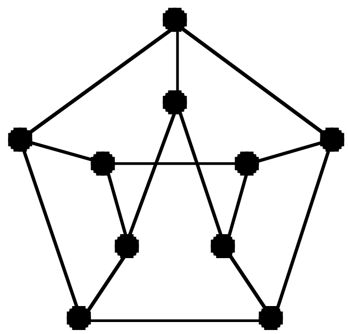
5b (10). Decide whether G and J are isomorphic. Explain.

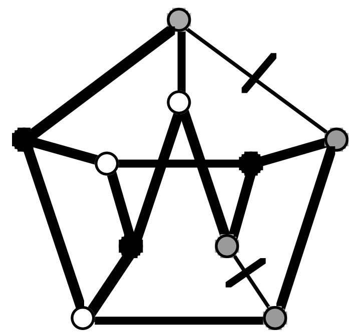
SOL (many ways): NO.
(1) diameter: diam (G) = 2; diam(J) = 3.
(2) girth: girth (G) = 5; J has 4-cycles.
(3) chromatic number: χ(G) = 3; χ(J) = 2 (bipartite).

5c (10). Decide whether H and J are isomorphic. Explain.

SOL (many ways): NO. but NOT due to diameter or girth.
(1) chromatic number: χ(H) = 3; χ(J) = 2 (bipartite).
(2) In J, but not in H, every edge lies in a 4-cycle.
(3) radius(H) = 2; radius(J) = 3.

6a (15). Mark two edges in the following graph such that it remains nonplanar after both are deleted. Prove your answer.





SOL: Draw subdivided $K_{3,3}$-subgraph as above, with two unused edges.

6b (15). Let G be the edge-complement of a 10-vertex, 3-regular simple graph. Prove that G is non-planar.

PROOF: Graph G must be 6-regular.
Thus, it has 30 edges.
The Euler polyhedral equation V-E+F = 2
implies that 10-30+F = 2. Thus, |F| = 22.
However, the edge-face inequality 2E ≥ 3F
implies that 60 = 2•30 ≥ 3•22 = 66, a contradiction!

7 (20). Consider the following recurrence:

$$a_0 = 0; \quad a_n = n^2 - na_{n-1}$$

7a (3). Calculate a_1, a_2, *and* a_3.

SOL: $a_1 = 1^2 - 1 \cdot a_0 = \underline{\underline{1}};$

$a_2 = 2^2 - 2 \cdot a_1 = 4 - 2 = \underline{\underline{2}};$

$a_3 = 3^2 - 3 \cdot a_2 = 9 - 6 = \underline{\underline{3}}.$

7c (17). Prove that $a_n = n$.

Proof by induction:

Basis: $a_0 = 0$.

Ind Hyp: *Assume that* $a_{n-1} = n - 1$.

Ind Step: $a_n = n^2 - n \cdot a_{n-1}$ *recursion*

$= n^2 - n \cdot (n-1)$ *by ind hyp*

$= n^2 - \left(n^2 - n\right) = n$

8 (25 pts). A bowl contains 50 fair coins {with p(H) = 0.5} and 50 standard loaded coins {with p(H) = 0.8}. A coin is drawn at random.

8a (10). The coin is tossed once. What is the probability of the outcome H?

$$SOL: \quad pr(H) = pr(H \wedge Fair) + pr(H \wedge Loaded)$$
$$= pr(H \mid Fair) \cdot pr(Fair) + pr(H \mid Loaded)pr(Loaded)$$
$$= 0.5 \cdot 0.5 + 0.8 \cdot 0.5 = 0.65$$

8b (15). If the outcome of the toss is H, what is the probability that the coin selected was loaded?

$$SOL: \quad pr(Loaded \mid H) = \frac{pr(Loaded \wedge H)}{pr(H)}$$
$$= \frac{0.4}{0.65} \ \ or \ \ \frac{8}{13}$$