14:38 10/15/2009

Chapter 8

Relations

8.1 RELATIONS & THEIR PROPERTIES

DEF: A ***binary relation*** is a predicate on the cartesian product $A \times B$ of two sets. Sometimes we say ***from*** A ***to*** B.

CONCEPTUALIZATION and MODELING

A binary relation is set-theoretically modeled as a subset of $A \times B$.

Example 8.1.1: $C(\cdot, \cdot)$ capital city of
domain $= C \times S$ cities and states
e.g., $C(\text{Albany, NY})$, $C(\text{Pierre, SD})$

Example 8.1.2: $\leq$ less than or equal to
domain $= \mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ = real numbers.
For instance, $\pi \leq 7.6$.

Conceptually, one usually regards "is capital of" and $\leq$ as

yes-no oracles

on ordered pairs. Their respective models as sets of ordered pairs are useful in representing these relations on a computer.

MORE EXAMPLES

Example 8.1.3: $E(\cdot,\cdot)$ eats
domain = $A \times A$, where A = animal species.
For instance, E(pythons, rabbits).

Example 8.1.4: $H(\cdot,\cdot)$ husband of
domain = $M \times F$ (males, females)
For instance, H(Jacob, Leah), H(Jacob, Rachel)

Example 8.1.5: $B(\cdot,\cdot)$ brother of
domain = $P \times P$, where P = all persons.
For instance, B(Joseph, Benjamin).

GENERALIZATION

Relations on products of more than two sets: ternary ("3-ary"), quarternary ("4-ary"), etc.
See §8.2 of Rosen text.

Example 8.1.6: R(a, e, y) means that a and e are father and mother, respectively, of child y.
Domain = (males, females, persons).
Thus, R(Abraham, Sarah, Isaac) is true.
N.B. R(Sarah, Abraham, Isaac) is meaningless and R(Isaac, Sarah, Abraham) is false.

ALTERNATIVE MODELS of RELATIONS

A binary relation can also be modeled as a list of lists of relatives or as a matrix.

Example 8.1.7: The relation Q from the set $\{1,2,3\}$ to the set $\{A,B,C\}$, with the ordered-pairs model

$$Q = \{(1,A),\ (1,B),\ (2,C),\ (3,A),\ (3,C)\}$$

has the ***lists-of-relatives model***

$$\begin{aligned} &1 : A, B \\ &2 : C \\ &3 : A, C \end{aligned}$$

and the ***matrix model***

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

COMPOSITION OF BINARY RELATIONS

DEF: Let Q be a relation from S to T and R a relation from T to U. Their ***composition*** $Q \circ R$ is the relation on $S \times U$ that is true for any pair (s, u) such that

$$(\exists t \in T)[Q(s,t) \wedge R(t,u)]$$

Remark: Using either the ordered-pairs model or the matrix model often makes the construction of compositions seem easier.

Example 8.1.8: Construct $Q \circ R$, where

$$Q = \{(1,A),\ (1,B),\ (2,C),\ (3,A),\ (3,C)\}$$

and

$$R = \{(A,x),\ (A,y),\ (A,z),\ (B,w),\ (B,y)\}$$

Using the definition of composition directly, we must consider every pair in the product

$$\{1,2,3\} \times \{w,x,y,z\}$$

and decide whether some member of $\{A, B, C\}$ relates the two coordinates.

CONTINUED

$$\underset{\displaystyle \left\{\begin{array}{l} 1: A, B \\ 2: C \\ 3: A, C \end{array}\right\}}{\underline{Q}} \circ \underset{\displaystyle \left\{\begin{array}{l} A: x, y, z \\ B: w, y \\ C: \emptyset \end{array}\right\}}{\underline{R}}$$

$$\underline{Q \circ R}$$

$$= \left\{\begin{array}{l} 1: w, x, y, z \\ 2: \emptyset \\ 3: x, y, z \end{array}\right\}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \times \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

DEF: Let R be a relation on set S. Then the powers of R are defined inductively:

$$R^0 = I \text{ (identity relation)} \qquad R^{n+1} = R^n \circ R$$

REFLEXIVE PROPERTY of RELATIONS

A binary relation R on a set S is ***reflexive*** if and only if

$$(\forall x \in S)\,[R\,(x, x)\,]$$

Example 8.1.2, continued: $\leq$
domain $= \mathcal{R} \times \mathcal{R}$, where $\mathcal{R}$ = real numbers.
REFLEXIVE, since $(\forall x \in \mathcal{R})[x \leq x]$.

Example 8.1.5, continued: $B(\cdot, \cdot)$ brother of
domain $= P \times P$, where P = all persons.
NONREFLEXIVE, since B(Joseph, Joseph) is false.

Fact. *A relation is reflexive if in its lists-of-relative model, every member of the domain is listed as one of its own relatives.*

Fact. *A relation is reflexive if its matrix model has 1's down the main diagonal.*

SYMMETRY PROPERTY of RELATIONS

A binary relation R on a set S is ***symmetric*** if and only if

$$(\forall x, y \in S)[R(x,y) \to R(y,x)]$$

Example 8.1.2, continued: $\leq$
domain $= \mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ = real numbers.
NONSYMMETRIC, since $\pi \leq 7$, but $7 \not\leq \pi$.

Example 8.1.5, continued: $B(\cdot,\cdot)$ brother of
domain $= P \times P$, where P = all persons.
QUESTION: If George is Bill's brother, does that imply that Bill is George's brother?

YES or NO

Example 8.1.3, continued: $E(\cdot,\cdot)$ eats
domain $= A \times A$, where A = animal species.
There are some symmetric pairs, such as ants and anteaters. Nonetheless, NONSYMMETRIC, since there also exist nonsymmetric pairs.

Fact. *A relation is symmetric iff*
$\forall x \forall y$ *[y occurs in the list of relatives of x*
$\rightarrow$ *x occurs in the list of relatives of y].*

Fact. *A relation is symmetric iff the relational matrix is symmetric around the main diagonal.*

Example 8.1.9: Some familial relationships are symmetric: spouse, sibling, cousin, in-law. Notice that none of them implies either an age difference or a gender.

Example 8.1.10: Some other familial relationships are non-symmetric: husband, sister, niece, parent. Each of them implies either an age difference or a gender.

CLASSROOM EXERCISE

Construct a 2-person domain in which step-grandfather-inlaw-hood is symmetric.

ANTISYMMETRY PROPERTY OF RELATIONS

A binary relation R is ***antisymmetric*** iff

$$(\forall x, y)[R(x, y) \wedge R(y, x) \ \rightarrow \ x = y]$$

Example 8.1.2, continued: $\leq$
domain $= \mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ = real numbers.
ANTISYMMETRIC, since

$$x \leq y \ \wedge \ y \leq x \ \Rightarrow \ x = y$$

Example 8.1.5, continued: $B(\cdot, \cdot)$ brother of
domain $= P \times P$, where P = all persons.
NON-ANTISYMMETRIC

Example 8.1.3, continued: $M(\cdot, \cdot)$ mother of
domain $= P \times P$, where P = all persons.
ANTISYMMETRIC, **vacuously**.

CLASSROOM EXERCISE

Let R be a relation that is both symmetric and antisymmetric. Prove that no element of its domain is related to any element other than possibly itself.

TRANSITIVITY PROPERTY OF RELATIONS

A binary relation R is ***transitive*** if and only if

$$(\forall x, y, z)[R(x, y) \wedge R(y, z) \rightarrow R(x, z)]$$

Example 8.1.1, continued: $\leq$
domain = $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ = real numbers.
TRANSITIVE, since

$$x \leq y \;\wedge\; y \leq z \;\Rightarrow\; x \leq z$$

Example 8.1.2, continued: $B(\cdot, \cdot)$ brother of
domain = $P \times P$, where P = all persons.
QUESTION: Is your brother's brother your brother?

YES or NO

Example 8.1.7: $M(\cdot, \cdot)$ mother of
domain = $P \times P$, where P = all persons.
NONTRANSITIVE, **vacuously**.

Example 8.1.8: *ancestor of*
domain = $P \times P$, where P = all persons.
TRANSITIVE

Prop 8.1.1. *Let R be a relation on a set S. Then R is transitive if and only if*

$$(\forall n \in \mathcal{Z}^{+})[R^n \subseteq R]$$

Pf: ($\Leftarrow$) $R^2 \subseteq R \;\Rightarrow\; R$ is transitive.

($\Rightarrow$) mathematical induction. $\diamondsuit$

8.3 REPRESENTING RELATIONS

A binary relation is conceptualized as a boolean-valued predicate on a cartesian product of two sets. We have already seen three combinatorial representations:

1. a set of ordered pairs
2. a list of lists-of-relatives
3. a matrix

We now introduce a conceptually powerful visual representation.

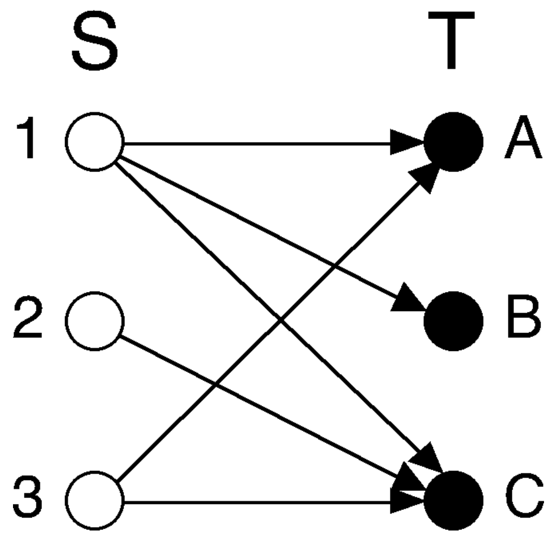
DEF: A ***digraph*** comprises a set whose elements are called ***vertices*** and a set whose elements are called ***arcs***. To each arc is associated a vertex called its ***head*** and a vertex called its ***tail***. We say the arc goes ***from*** its tail ***to*** its head.

DEF: The ***digraph representation*** of a binary relation R from a set S to a set T has $S \cup T$ as its vertex set. For $x, y \in S$, there is an arc from x to y if $R(x, y)$.

When $S \neq T$, the digraph is often drawn so that set S is represented in the left column and T in the right column.

Example 8.1.7, continued:

$$Q = \{(1, A),\ (1, B),\ (2, C),\ (3, A),\ (3, C)\}$$

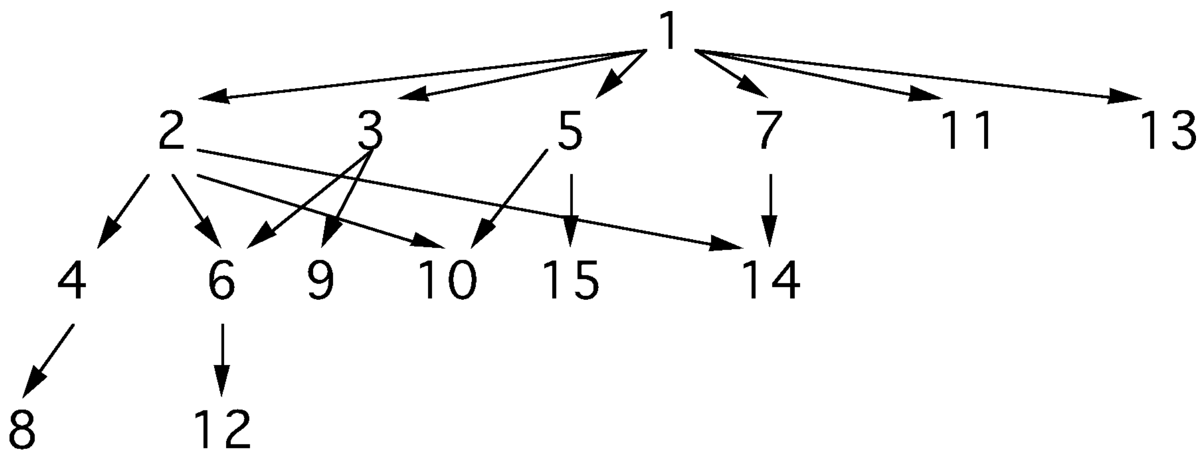


If the relation is from a set S to itself, then the elements of S are placed in conceptually meaningful locations.

Example 8.3.1: Let $S = \{1, 2, \ldots, 14\}$, with $R(x, y)$ if and only if these two conditions hold:

(1) x properly divides y.

(2) There is no number u such that x properly divides z and u properly divides y.



The relation of proper divisibility is not reflexive;

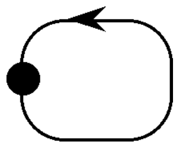
it is not symmetric;

it is not transitive;

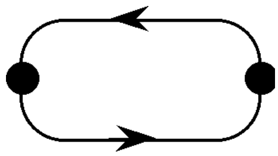
however, it is anti-transitive.

DIGRAPHS and RELATIONAL PROPERTIES

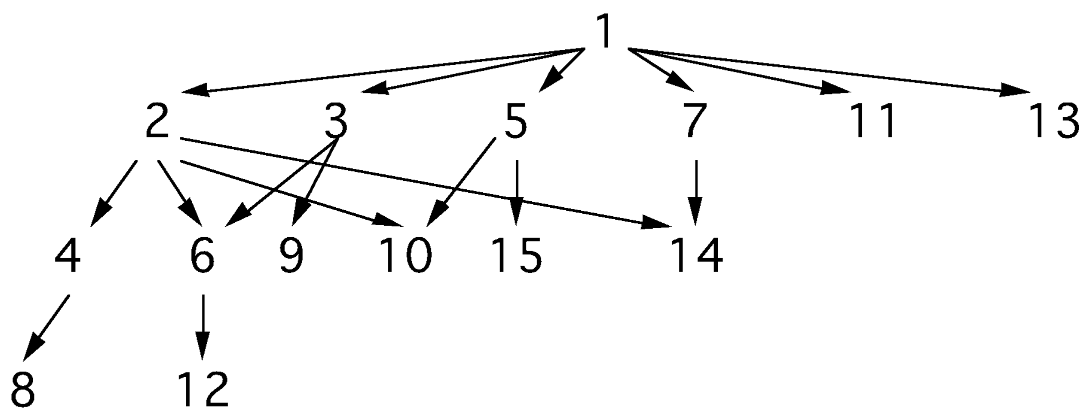
Prop 8.3.1. *A relation R is reflexive iff there is a self-loop at every vertex of its digraph.*



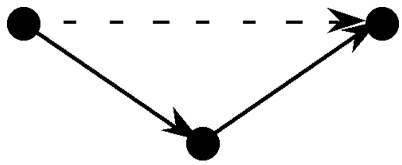
Prop 8.3.2. *A relation R is symmetric iff whenever there is an arc from x to y in its digraph, there is also an arc from y to x.*



Prop 8.3.3. *A relation R is antisymmetric iff whenever there is an arc from x to y in its digraph, with $x \neq y$, there is no arc from y to x.*



Prop 8.3.4. *A relation R is transitive iff whenever there is a directed path from x to y in its digraph, there is also an arc directly from x to y.*



Example 8.3.1, continued: Compare these graphics to the matrix representation.

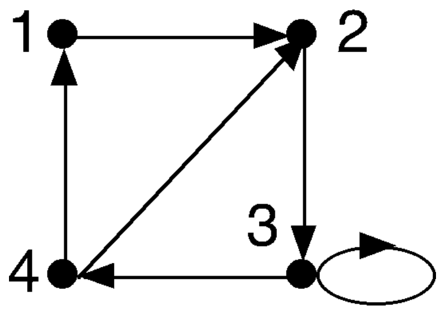
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	1	1	0	1	0	1	0	0	0	1	0	1	0
2	0	0	0	1	0	1	0	0	0	1	0	0	0	1
3	0	0	0	0	0	1	0	0	1	0	0	0	0	0
4	0	0	0	0	0	0	0	1	0	0	0	1	0	0
5	0	0	0	0	0	0	0	0	0	1	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	1	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	1
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0

COMPUTING R^{n} WITH A DIGRAPH

To calculate R^n from the digraph for R, simply list the ordered endpoint pairs for each directed path of length n.

Example 8.3.2: Consider this relation:

$$R = \{(1,2),(2,3),(3,3),(3,4),(4,1)\}$$



$$R^2 = \{(1,3),(2,3),(2,4),(3,1), (3,3),(3,4),(4,2)\}$$

$$R^3 = \{(1,3),(1,4), (2,1),(2,3),(2,4), (3,1),(3,2),(3,3),(3,4), (4,3),(4,4)\}$$

8.4 CLOSURES OF RELATIONS

DEF: The ***closure of a relation*** R with respect to a property is the intersection of all relations R' containing R.

Prop 8.4.1. *The reflexive closure of a relation R on a set S is the union $R \cup \{(s,s)|s \in S\}$.* $\diamondsuit$

Example 8.4.1: The reflexive closure of

the relation *sibling of*

is the relation *has the same two parents.*

Example 8.4.2: The reflexive closure of proper divisibility is divisibility.

The reflexive closure of a relation could be represented digraphically be drawing a self-loop at each vertex that did not already have one.

In the matrix representation, one could write 1's down the main diagonal.

Prop 8.4.2. *The symmetric closure of a relation R is the relation $R \cup R^{-1}$.*

CLASSROOM EXERCISE

Is siblinghood the symmetric closure of brotherhood?

Prop 8.4.3. *The transitive closure of a relation R on a set S is the relation*

$$R^* = \bigcup_{j=1}^{\infty} R^j$$

Example 8.4.3: the relation *parent of*
The transitive closure is *proper ancestor of.*
The reflexive, transitive closure is *ancestor of.*

CLASSROOM EXERCISES

Q1. What is the antisymmetric closure of brotherhood?

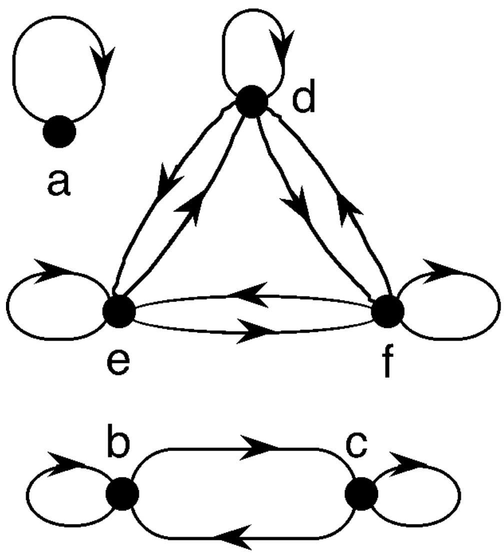
Q2. To whom are you related under the transitive closure of the symmetric closure of parenthood?

8.5 EQUIVALENCE RELATIONS

DEF: An ***equivalence relation*** is a binary relation that is reflexive, symmetric, and transitive.

Example 8.5.1: Set $S = \{a, b, c, d, e, f\}$ and

$$R = \{\ (a,a),(b,b),(b,c),(c,b),(c,c),(d,d),(d,e),\\ (d,f),(e,d),(e,e),(e,f),(f,d),(f,e),(f,f)\ \}$$



	a	b	c	d	e	f
a	1					
b		1	1			
c		1	1			
d				1	1	1
e				1	1	1
f				1	1	1

Prop 8.5.1. *Let R be an equivalence relation. Then every component of the digraph of R is a complete digraph.* $\diamondsuit$

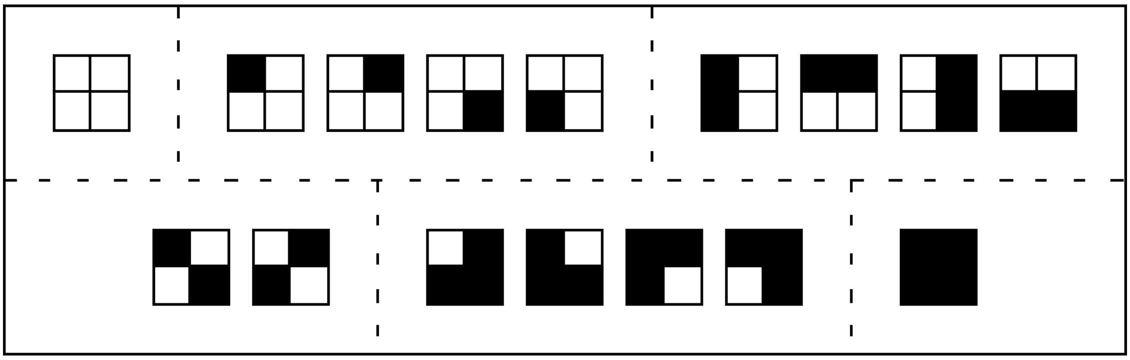
Cor 8.5.2. *An equivalence relation induces a partition on its domain.*

Pf: The vertex set of each component of the digraph is a cell of the partition. $\diamondsuit$

Prop 8.5.3. *Let R be an equivalence relation. Order its domain so that the elements of each cell of the induced partition occur contiguously. The resulting matrix represention has blocks of square matrices of all 1's down its main diagonal and has all zeroes for its other entries.* $\diamondsuit$

FINITE EQUIVALENCE RELATIONS

Example 8.5.2: 2×2 checkerboards



There are 16 checkerboards. Checkerboard x is related to checkerboard y if it can be transformed into y by a rotation or by a reflection.

Example 8.5.3: relation $R =$ *sibling of*
domain = all persons

The reflexive, trans. closure of R is an eq rel. It partitions all of humanity into equivalence classes of (full) siblings.

INFINITE EQUIVALENCE RELATIONS

Example 8.5.4: domain = rational fractions

$$\left\{ \frac{p}{q} \;\middle|\; p, q \in \mathcal{Z}, q \neq 0 \right\}$$

$$\text{Then} \quad \frac{a}{b} \text{ and } \frac{c}{d} \quad \text{are related if} \quad ad = bc$$

The partition cells are rational fractions of equal value.

COUNTING PROBLEM (solved in w4205): How many cells are there if $p, q \in \{1, \ldots, 10\}$?

Example 8.5.5: domain $\mathcal{Z}$
eq. rel. = congruence mod 3
Equivalence Classes:

$$[0]_3 = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$$
$$[1]_3 = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$$
$$[2]_3 = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$$

Example 8.5.6: domain = propositions on p, q
(infinite domain – arbitrarily long strings)
eq. rel. = logical equivalence

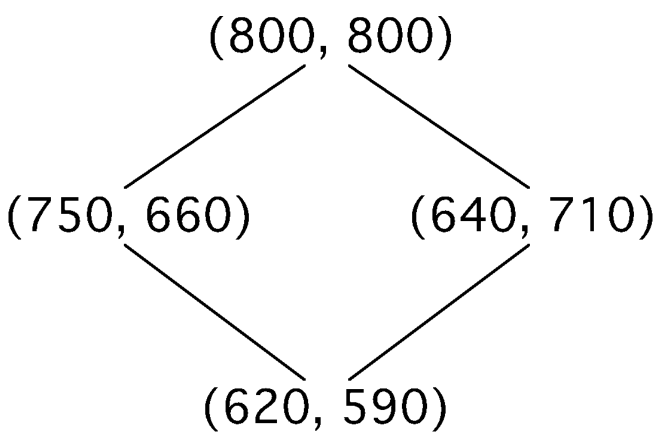
N.B. there are 2^4 cells to this partition

8.6 PARTIAL ORDERINGS

DEF: A ***partial ordering*** is a binary relation $\preceq$ that is reflexive, antisymmetric, and transitive.

DEF: A ***partially ordered set*** or ***poset*** is a pair $\langle S, \preceq \rangle$ consisting of a set and a partial ordering on that set.

Example 8.6.1: domain: SAT scores (M, V)
relation: double domination

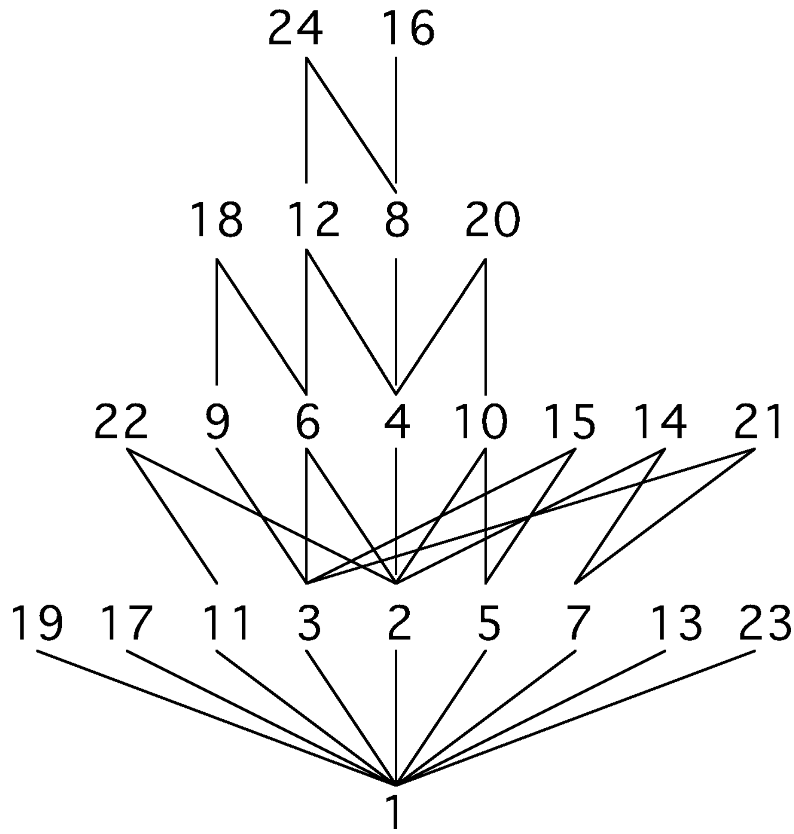


NOTATION: We write $x \prec y$ if $x \preceq y$ and $x \neq y$.

DEF: An element y ***covers*** an element x in a poset if $x \prec y$ and $(\nexists u)[x \prec u \prec y]$.

DEF: In a ***Hasse diagram*** for a poset, one draws only the cover relations.

Example 8.6.2: $(\mathcal{Z}^+, \backslash)$ the division lattice*
Special Case: $\{n \mid 1 \leq n \leq 24\}$



Fact. *The transitive, reflexive closure of the cover relationship is the poset itself.*

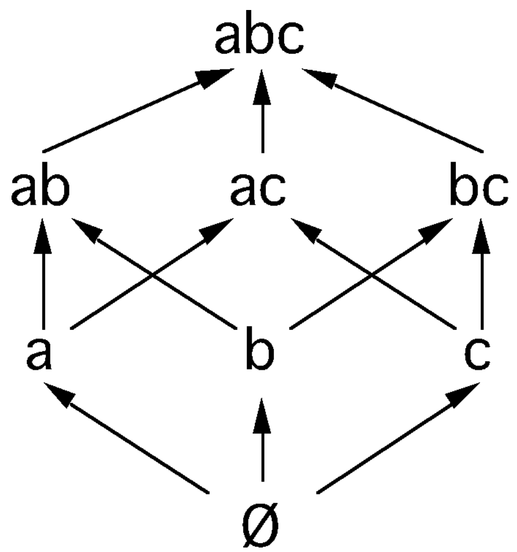
Prop 8.6.1. *Divisibility of positive integers is a partial ordering.*

Pf:
Reflexive: a number divides itself.

Antisymmetric: if $x \backslash y$ and $y \backslash x$, then $x = y$.

Transitive. if $x \backslash y$ and $y \backslash z$, then $x \backslash z$. $\diamondsuit$

Example 8.6.3: $(X, \subseteq)$ the subset lattice
Special Case: $X = \{a, b, c\}$



Reflexive: $S \subseteq S$.

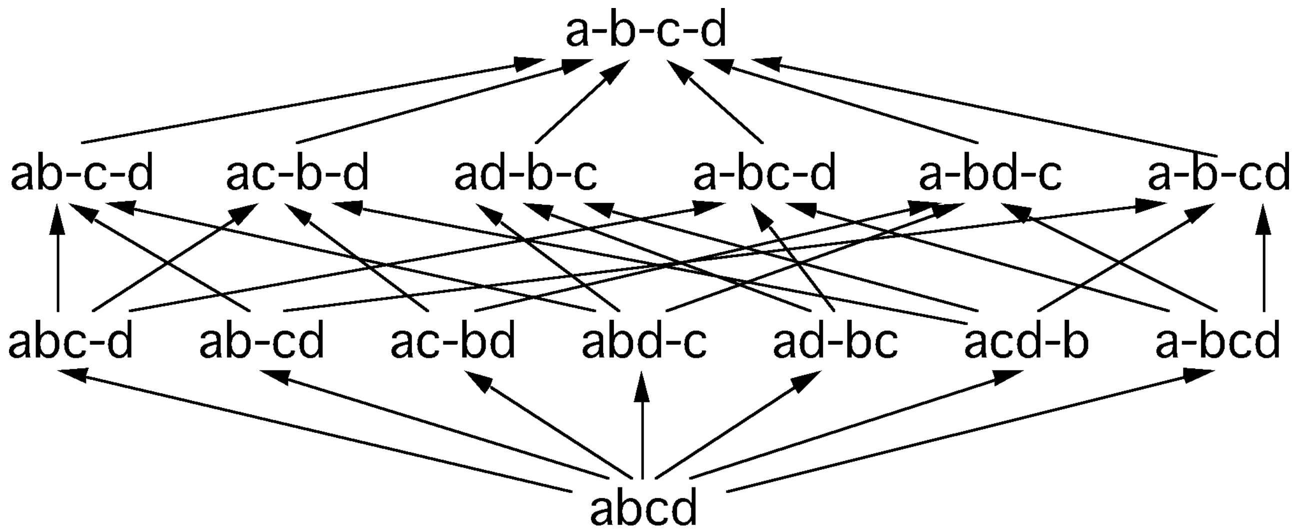
Antisymmetric: if $S \subseteq T$ and $T \subseteq S$ then $S = T$.

Transitive. if $S \subseteq T$ and $T \subseteq U$ then $S \subseteq U$.

* a "lattice" is a special kind of poset.

DEF: A partition P of a set S ***refines*** another partition S if every cell of P is a subset of some cell of Q.

Example 8.6.4: the refinement lattice
domain: partitions of a set S
relation: inverse refinement *is refined by*
Special Case: $S = \{a, b, c, d\}$



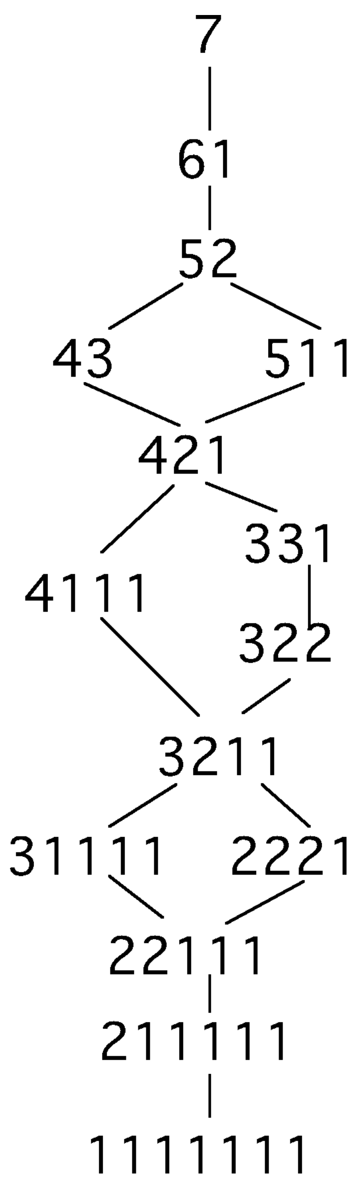
DEF: Let $P = (p_1, \ldots, p_j)$ and $Q = (q_1, ..., q_k)$ be two partitions of the same integer. ***Summation dominance*** is the relation

$$P = (p_1, \ldots, p_j) \ \leq \ Q = (q_1, ..., q_k)$$

if and only if

$$(\forall t \geq 1) \, [p_1 + \cdots + p_t \ \leq \ q_1 + \cdots + q_t]$$

Example 8.6.5: summation dominance lattice
domain: partitions of an integer n
relation: summation dominance
Special Case: $n = 7$.



INCOMPARABILITY

DEF: Two elements x, y from a poset $\langle X, R \rangle$ are said to be ***comparable*** if either xRy or yRx and ***incomparable*** otherwise.

Example 8.6.1, continued: College Bd Scores under double domination. Incomparable pairs:

$$(500, 700) \quad \text{and} \quad (730, 580)$$

Example 8.6.2, continued: division lattice Incomparable integers:

$$9 \quad \text{and} \quad 20$$

Example 8.6.3, continued: subset lattice Incomparable subsets:

$$\{a, b\} \quad \text{and} \quad \{b, c\}$$

Example 8.6.4, continued: ref'ment lattice Incomparable partitions:

abc-*d* and *ab*-*cd*

Example 8.6.5, continued: sum dominance Incomparable partitions:

$$4111 \quad \text{and} \quad 322$$

TOTAL ORDERINGS and LINEAR EXTENSIONS

DEF: A partial ordering is called ***total, complete,*** or ***linear*** if every pair of elements is comparable.

Example 8.6.6: $(\mathcal{R}, \leq)$

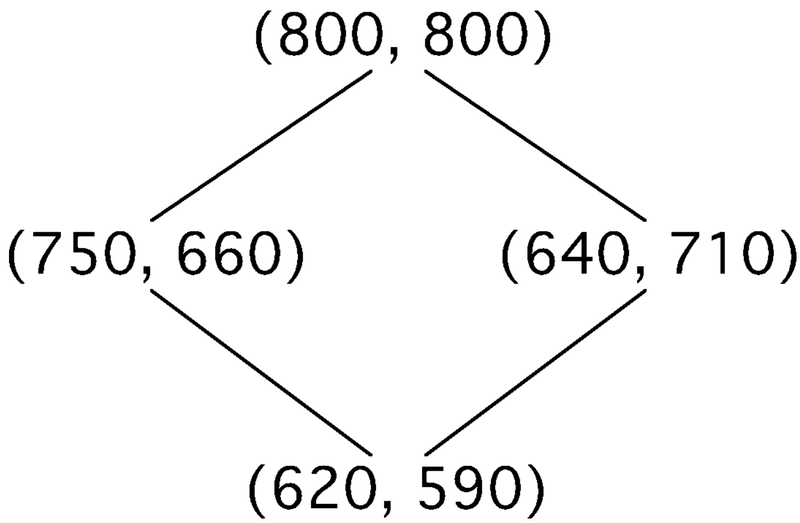
Example 8.6.7: alphabetic strings under lexicographic ordering.

DEF: A ***linear extension*** of a partial ordering R on a set S is a total ordering Q on S such that $R \subseteq Q$.

Thm 8.6.2. *Every partial ordering R on a finite set S has a linear extension.*

Pf: Remove any minimal element and put it first. Then follow it inductively by a linear extension of the rest of the poset. $\diamondsuit$

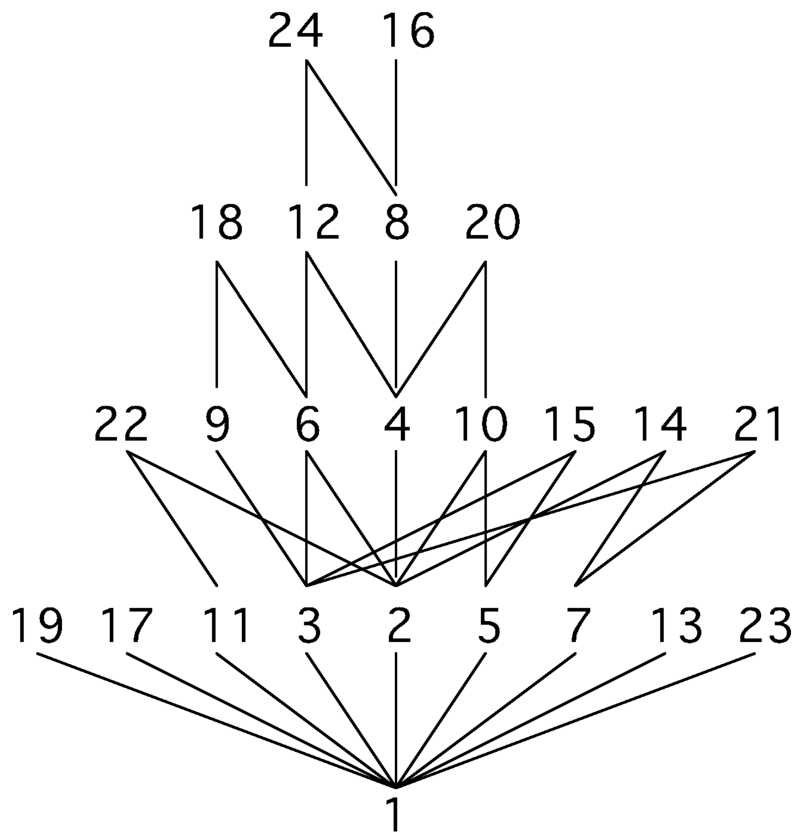
Example 8.6.1, continued: SAT scores relation: double domination



linear extensions:

$$(620, 590) \leq (750, 660) \leq (640, 710) \leq (800, 800)$$
$$(620, 590) \leq (640, 710) \leq (750, 660) \leq (800, 800)$$

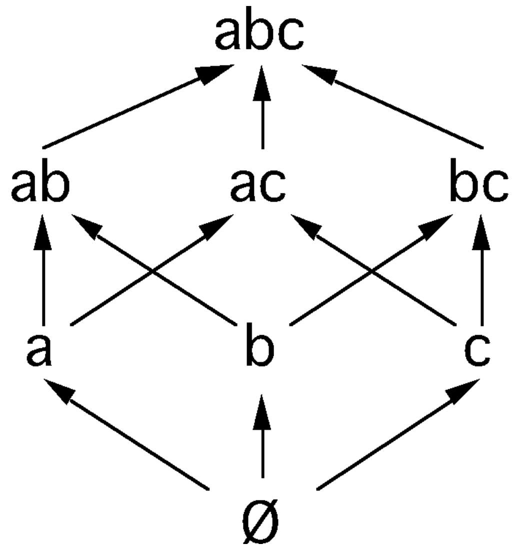
Example 8.6.2, continued: division lattice
Special Case: $\{n \mid 1 \leq n \leq 24\}$



linear extension:

$$
\begin{aligned}
&1 \leq 2 \leq 3 \leq 5 \leq 7 \leq 11 \leq 13 \leq 17 \leq 19 \leq 23 \\
&\quad \leq 4 \leq 6 \leq 9 \leq 10 \leq 14 \leq 15 \leq 21 \leq 22 \\
&\quad \leq 8 \leq 12 \leq 18 \leq 20 \\
&\quad \leq 16 \leq 24
\end{aligned}
$$

Example 8.6.3, continued: subset lattice
Special Case: $X = \{a, b, c\}$



linear extensions:

$$\emptyset \leq a \leq b \leq c \leq ab \leq ac \leq bc \leq abc$$
$$\emptyset \leq a \leq b \leq ab \leq c \leq ac \leq bc \leq abc$$

An algorithm that arranges the elements of a poset into such a sequence is sometimes called a ***topological sort***.

Remark: Most familiar sorting algorithms can be readily adapted to the task of topological sorting.

Remark: A "topological sort" has almost nothing to do with topology.

LUB's GLB's and LATTICES

DEF: An ***upper bound*** in a poset $\langle S, \preceq \rangle$ for a subset $T \subseteq S$ is an element $u \in S$ such that

$$(\forall t \in T)\,[t \preceq u]$$

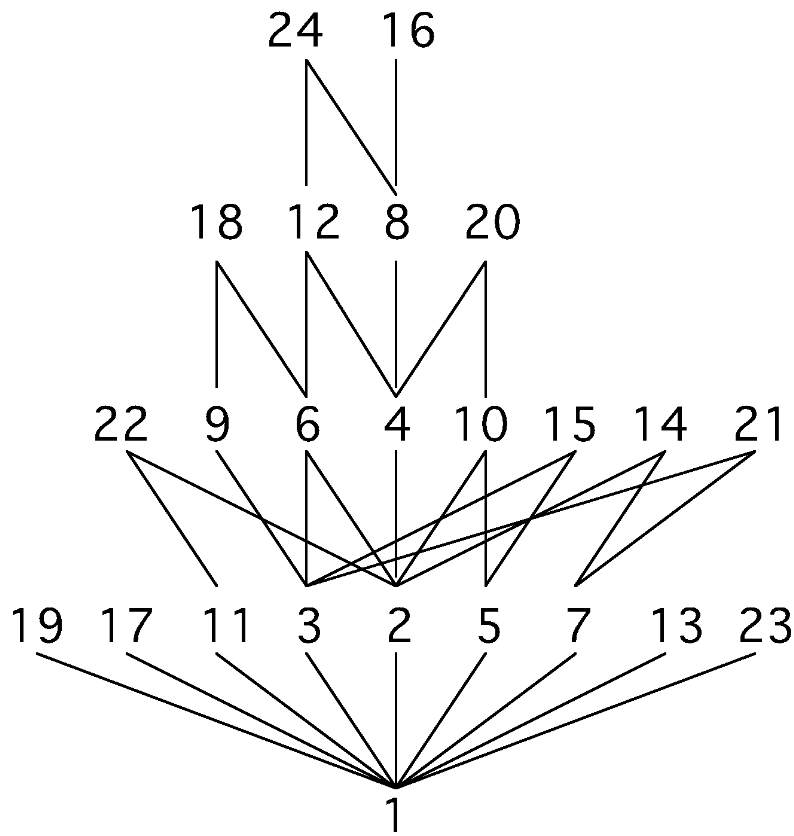
A ***least upper bound*** (abbr. ***lub***) for a subset T of a poset $\langle S, \preceq \rangle$ is an upper bound u such that if v is another upper bound, then $u \preceq v$.

DEF: A ***lower bound*** in a poset $\langle S, \preceq \rangle$ for a subset $T \subseteq S$ is an element $u \in S$ such that

$$(\forall t \in T)\,[u \preceq t]$$

A ***greatest lower bound*** (abbr. ***glb***) for a subset T of a poset $\langle S, \preceq \rangle$ is a lower bound u such that if v is another lower bound, then $v \preceq u$.

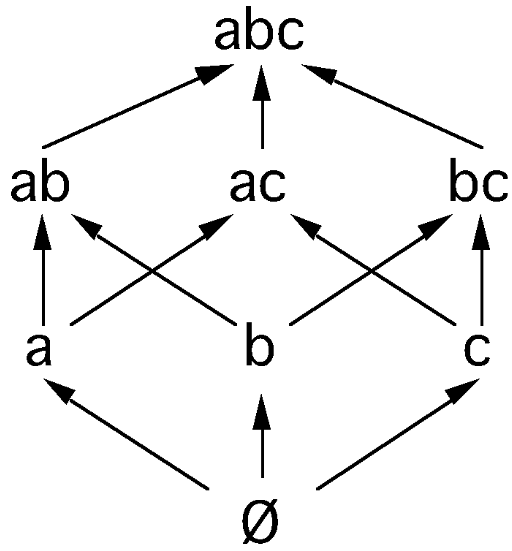
Example 8.6.2, continued: division lattice
Special Case: $\{n \mid 1 \leq n \leq 24\}$



$\mathrm{LUB}(x, y) = \mathrm{lcm}(x, y).$

$\mathrm{GLB}(x, y) = \mathrm{gcd}(x, y).$

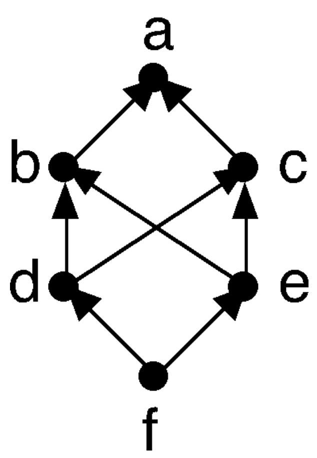
Example 8.6.3, continued: subset lattice
Special Case: $X = \{a, b, c\}$



$\mathrm{LUB}(S, T) = S \cup T.$

$\mathrm{GLB}(S, T) = S \cap T.$

Example 8.6.4: NOT a lattice



b and c have no glb

d and e have no lub