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Chapter 7

Advanced Counting

coursenotes by Prof. J. L. Gross for Rosen 6th Edition

7.1 RECURRENCE RELATIONS

DEF: A ***recurrence system*** is a finite set of ***initial conditions***

$$a_0 = c_0, \quad a_1 = c_1, \quad \ldots, \quad a_d = c_d$$

and a formula (called a ***recurrence relation***)

$$a_n \;=\; f(a_0, \ldots, a_{n-1})$$

that expresses a subscripted variable as a function of lower-indexed values. A sequence

$$< a_n > = \; a_0,\; a_1,\; a_2,\; \ldots$$

satisfying the initial conditions and the recurrence relation is called a ***solution***.

Example 7.1.1: The recurrence system

$$\begin{array}{rcll} a_0 & = & 0 & \text{initial condition} \\ a_n & = & a_{n-1} + 2n - 1 & \text{recurrence relation} \end{array}$$

has the sequence of squares as its solution:

$$< a_n > \;=\; 0,\; 1,\; 4,\; 9,\; 16,\; 25,\; \ldots$$

NAIVE METHOD OF SOLUTION

Step 1. Use the recurrence to calculate a few more values beyond the given initial values.

Step 2. Spot a pattern and guess the right answer.

Step 3. Prove your answer is correct (by induction).

Example 7.1.1, continued:
Step 1. Starting from $a_0 = 0$, we calculate

$$\begin{aligned} a_1 &= a_0 + 2 \cdot 1 - 1 = 0 + 1 = 1 \\ a_2 &= a_1 + 2 \cdot 2 - 1 = 1 + 3 = 4 \\ a_1 &= a_0 + 2 \cdot 3 - 1 = 4 + 5 = 9 \\ a_1 &= a_0 + 2 \cdot 4 - 1 = 9 + 7 = 16 \end{aligned}$$

Step 2. Looks like $f(n) = n^2$.

Step 3. BASIS: $a_0 = 0 = 0^2 = f(0)$.

IND HYP: Assume that $a_{n-1} = (n-1)^2$.

IND STEP: Then

$$\begin{aligned} a_n &= a_{n-1} + 2n - 1 && \text{from the recursion} \\ &= (n-1)^2 + 2n - 1 && \text{by IND HYP} \\ &= (n^2 - 2n + 1) + 2n - 1 = n^2 && \diamondsuit \end{aligned}$$

APPLICATIONS

Example 7.1.2: Compound Interest
Deposit \$1 to compound at annual rate r.
$p_0 = 1 \quad p_n = (1+r)p_{n-1}$

EARLY TERMS: $1, 1+r, (1+r)^2, (1+r)^3, \ldots$

APPARENT PATTERN: $p_n = (1+r)^n$

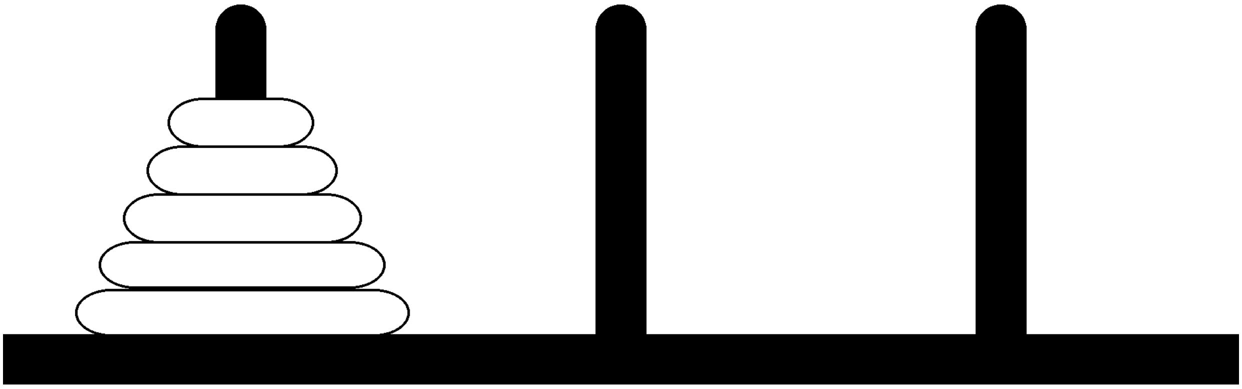
Pf: BASIS: True for n = 0.

IND HYP: Assume that $p_{n-1} = (1+r)^{n-1}$

IND STEP: Then

$$\begin{aligned} p_n &= (1+r)p_{n-1} \quad \text{by the recursion} \\ &= (1+r)(1+r)^{n-1} \quad \text{by IND HYP} \\ &= (1+r)^n \quad \text{by arithmetic} \end{aligned}$$
$\diamondsuit$

Example 7.1.3: Tower of Hanoi



RECURRENCE SYSTEM

$$h_0 = 0$$
$$h_n = 2h_{n-1} + 1$$

SMALL CASES: $0,\ 1,\ 3,\ 7,\ 15,\ 31,\ \ldots$

APPARENT PATTERN: $h_n = 2^n - 1$

BASIS: $h_0 = 0 = 2^0 - 1$

IND HYP: Assume that $h_{n-1} = 2^{n-1} - 1$

IND STEP: Then

$$\begin{aligned} h_n &= 2h_{n-1} + 1 \quad \text{by the recursion} \\ &= 2(2^{n-1} - 1) + 1 \quad \text{by IND HYP} \\ &= 2^n - 1 \quad \text{by arithmetic} \end{aligned}$$ ♢

However, the näive method has limitations:

- It can be non-trivial to spot the pattern.
- It can be non-trivial to prove that the apparent pattern is correct.

Example 7.1.4: Fibonacci Numbers

$$f_0 = 0 \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

Fibo seq: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$.

APPARENT PATTERN (ha ha)

$$f_n = \frac{1}{2^n\sqrt{5}} \left[(1+\sqrt{5})^n - (1-\sqrt{5})^n \right]$$

It is possible, but not uncomplicated, to simplify this with the binomial expansion and to then use induction.

Sometimes there is no fixed limit on the number of previous terms used by a recursion.

Example 7.1.5: Catalan Recursion

$$\begin{aligned} c_0 &= 1 \\ c_n &= c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0 \quad \text{for } n \geq 1 \end{aligned}$$

SMALL CASES

$$\begin{aligned} c_1 &= c_0 c_0 = 1 \cdot 1 = 1 \\ c_2 &= c_0 c_1 + c_1 c_0 = 1 \cdot 1 + 1 \cdot 1 = 2 \\ c_3 &= c_0 c_2 + c_1 c_1 + c_2 c_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5 \\ c_4 &= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14 \\ c_5 &= 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42 \end{aligned}$$

Catalan seq: 1, 1, 2, 5, 14, 42, ...

SOLUTION: $c_n = \dfrac{1}{n+1}\dbinom{2n}{n}$

The Catalan recursion counts binary trees and other objects in computer science.

ADMONITION

- Most recurrence relations have no solution.
- Most sequences have no representation as a recurrence relation. (they are random)

7.2 SOLVING EASY RECURRENCES

We identify a type of recurrence that can be solved by special methods.

DEF: A recurrence relation

$$a_n = f(a_0, \ldots, a_{n-1})$$

has ***degree*** k if the function f depends on the term a_{n-k} and if it depends on no terms of lower index. It is ***linear of degree*** k if it has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + g(n)$$

where each c_k is a real function and $c_k \neq 0$. It is ***homogenous*** if $g(n) = 0$.

Example 7.2.1: The recurrence system

$$\begin{aligned} a_0 &= 0 && \text{initial condition} \\ a_n &= a_{n-1} + 2n - 1 && \text{recurrence relation} \end{aligned}$$

is linear of degree one and non-homogeneous.

Remark: Similarly, the interest recursion and the Tower of Hanoi recursion are linear of degree one and non-homogeneous.

Example 7.2.2: Fibonacci Numbers

$$\begin{aligned} f_0 &= 1 \qquad f_1 = 1 && \text{initial conditions} \\ f_n &= f_{n-1} + f_{n-2} && \text{recurrence relation} \end{aligned}$$

The Fibonacci recurrence is linear of degree two and homogeneous.

Example 7.2.3: Catalan Recursion

$$\begin{aligned} c_0 &= 1 && \text{initial condition} \\ c_n &= c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0 && \text{for } n \geq 1 \end{aligned}$$

The Catalan recusion is quadratic, homogeneous, and not of fixed degree.

Remark: Solving the Catalan recursion is well beyond the level of this course.

HOMOG LINEAR RR's w. CONST COEFF'S

DEF: The ***special method*** for solving an homogeneous linear RR with constant coeff's

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is as follows:

1. Assume there exists a solution of the form

$$a_n = r^n$$

and substitute it into the recurrence:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

Cancelling the excess powers of r and normalizing yields what is called the ***characteristic equation:***

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

2. Find the roots of the char eq,

$$r_1, r_2, \ldots, r_k$$

which are called the ***characteristic roots.***

3. Form the ***general solution***

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

4. Use initial conditions to form k simultaneous linear equations in $\alpha_1, \ldots, \alpha_k$ and solve for them.

DEGREE ONE, LINEAR HOMOGENEOUS

Example 7.2.4: General RR of Degree 1

$$\begin{aligned} a_0 &= d & \text{initial condition} \\ a_n &= ca^{n-1} & \text{recurrence relation} \end{aligned}$$

char eq: $r - c \;=\; 0$ has root $r = c$

general solution: $a_n = \alpha_1 c^n$

simultaneous linear equations: $d = \alpha_1 c^0 = \alpha_1$

solution to simult lin eq: $\alpha_1 = d$

problem solution: $a_n = dc^n$

Example 7.2.5: Compound Interest again

Deposit \$3 to be compounded annually at rate r.

$$\begin{aligned} p_0 &= 3 \\ p_n &= (1 + r)\, p_{n-1} \end{aligned}$$

Solution: $p_n = 3(1 + r)^n$

DEGREE TWO, LINEAR HOMOGENEOUS

Example 7.2.6: Easy degree two recurrence.

$$a_0 = 1 \qquad a_1 = 4$$
$$a_n = 5a_{n-1} - 6a_{n-2}$$

char eq: $r^2 - 5r + 6 = 0$ has roots $r_1 = 3 \quad r_2 = 2$.
gen sol: $a_n = \alpha_1 3^n + \alpha_2 2^n$

simult lin eqns $\begin{cases} a_0 = 1 = \alpha_1 + \alpha_2 \\ a_1 = 4 = 3\alpha_1 + 2\alpha_2 \end{cases}$

have solution: $\alpha_1 = 2 \quad \alpha_2 = -1$.
$\Rightarrow$ problem solution: $a_n = 2 \cdot 3^n - 2^n$

Changing the initial conditions to

$$a_0 = 2 \quad a_1 = 5$$

yields

simult lin eqns $\begin{cases} a_0 = 2 = \alpha_1 + \alpha_2 \\ a_1 = 5 = 3\alpha_1 + 2\alpha_2 \end{cases}$

with solution: $\alpha_1 = 1 \quad \alpha_2 = 1$.
$\Rightarrow$ problem solution: $a_n = 3^n + 2^n$

Example 7.2.7: Fibonacci Numbers again

$$f_0 = 0 \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

char eq: $r^2 - r - 1 = 0$ has roots

$$\frac{1+\sqrt{5}}{2} \text{ and } \frac{1-\sqrt{5}}{2}$$

Etc. The complete solution is

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \frac{1}{2^n\sqrt{5}}\left[(1+\sqrt{5}\,)^n - (1-\sqrt{5}\,)^n\right] \\ &= \frac{1}{2^n}\sum_{j=0}^{\lfloor n/2 \rfloor}\binom{n}{2j+1}\left(\sqrt{5}\right)^{2j} \end{aligned}$$

For instance,

$$\begin{aligned} f_5 &= \frac{1}{16}\left[\binom{5}{1} + \binom{5}{3}5 + \binom{5}{5}5^2\right] \\ &= \frac{1}{16}[5 + 50 + 25] = \frac{80}{16} = 5 \end{aligned}$$

DEGREE THREE, LINEAR HOMOGENEOUS

Example 7.2.8:

$$a_0 = 2 \qquad a_1 = 5 \qquad a_2 = 15$$
$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

char eq:

$$0 = r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$$

char roots:

$$r = 1, 2, 3$$

gen sol:

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$$

simult lin eq:

$$\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3 \\ a_1 &= 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3 \\ a_2 &= 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9 \end{aligned}$$

coeff solns:

$$\alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = 2$$

unique sol:

$$a_n = 1 - 2^n + 2 \cdot 3^n$$

NONHOMOGENEOUS LINEAR RECURRENCES

We split the solution into a homogeneous part and a particular part.

Example 7.2.9: Tower of Hanoi, again

$$\begin{aligned} h_0 &= 0 \\ h_n &= 2h_{n-1} + 1 \end{aligned}$$

assoc homog relation $\hat{h}_n = 2\hat{h}_{n-1}$ has

homogeneous solution $\hat{h}_n = \alpha 2^n$

assoc partic relation $\dot{h}_n = 2\dot{h}_{n-1} + 1$ has

particular solution $\dot{h}_n = -1$

simult lin eqn:

$$h_0 = 0 = \hat{h}_0 + \dot{h}_0 = \alpha 2^0 - 1$$

has solution $\alpha = 1$

problem solution: $h_n = 2^n - 1$.

Remark: The form of the particular solution usually resembles the function of n. In this case

$$g(n) = 1$$

is a constant function. So we tried $\dot{h}_n = K$, and we solved the equation $K = 2K + 1$, and obtained $K = -1$.

Example 7.2.10:

$$\begin{aligned} a_1 &= 3 \\ a_n &= 3a_{n-1} + 2n \end{aligned}$$

homog soln:

$\hat{a}_n = \alpha 3^n$

partic rec rel:

$\hat{a}_n = 3\hat{a}_{n-1} + 2n$

trial soln:

$\hat{a}_n = cn + d$

Then $cn + d = 3[c(n-1) + d] + 2n$,
i.e., $0 = n(2c+2) + (2d - 3c)$
$\Rightarrow \; c = -1, \quad d = -3/2$

partic soln:

$\hat{a}_n = -n - 3/2$

general soln:

$a_n = \alpha 3^n - n - 3/2$

simult eq:

$a_1 = 3 = \alpha 3 - 1 - 3/2 = 3\alpha - 5/2$

coeff solns:

$\alpha = 11/6$

unique sol:

$$a_n = \frac{11}{6} 3^n - n - \frac{3}{2}$$

REPEATED ROOTS

Example 7.2.11: A recurrence system

$$a_0 = -2 \qquad a_1 = 2$$

$$a_n = 4a_{n-1} - 4a_{n-2}$$

char eq: $r^2 - 4r + 4 = 0$ has roots 2, 2.
gen sol: $a_n = \alpha_1 2^n + \alpha_2 n 2^n$

simult lin eqns $\begin{cases} a_0 = -2 = \alpha_1 \\ a_1 = 2 = 2\alpha_1 + 2\alpha_2 \end{cases}$

have solution: $\alpha_1 = -2 \qquad \alpha_2 = 3.$
problem solution: $a_n = (-2) \cdot 2^n + 3 \cdot n2^n$

7.3 DIVIDE-AND-CONQUER RELATIONS

The bag-of-tricks used for recurrence relations is really quite deep.

DEF: A ***d***ivide-and-conquer recurrence has the following form:

$$f(n) = af(\frac{n}{b}) + g(n)$$

Thm 7.3.1. *A divide-and-conquer recurrence*

$$f(n) = af(\frac{n}{b}) + c$$

in which b is a positive integer and $c > 0$ has the following property:

$$f(n) \in \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1 \end{cases}$$

Example 7.3.1: Binary Search can be evaluated by Thm 7.3.1.

$$\begin{aligned} b_1 &= 1 \\ b_n &= b_{n/2} + 1 \end{aligned}$$

Then $b_n \in O(\log n)$ by Thm 7.3.1.

Example 7.3.2: MergeSort needs a different approach, because its nonhomogenous term is not a constant.

$$\begin{aligned} s_1 &= 0 \\ s_n &= 2s_{n/2} + n \quad \text{for } n > 1 \end{aligned}$$

The substitutions

$$n = 2^k \quad \text{and} \quad t_k = s_{2^k}$$

change this divide-and-conquer recurrence to the following linear recurrence:

$$\begin{aligned} t_0 &= 0 \\ t_k &= 2t_{k-1} + 2^k \quad \text{for } k > 0 \end{aligned}$$

sol: $t_k = k \cdot 2^k$

Then reverse-substitute to obtain

sol: $s_n = n \lg n$.

7.4 GENERATING FUNCTIONS

Any sequence $a_0, a_1, a_2, \ldots$ can be encoded as a formal infinite power series

$$a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$$

DEF: An ***(ordinary) generating function*** for a sequence $a_0, a_1, a_2, \ldots$ is a function whose formal power series (Maclaurin series) has that sequence as its sequence of coefficients.

Example 7.4.1: The generating function for the sequence $1, 1, 1, \ldots$ is the function

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{j=0}^{\infty} x^j$$

Example 7.4.2: The generating function for the sequence $1, 2, 4, 8, \ldots$ is the function

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots = \sum_{j=0}^{\infty} 2^j x^j$$

RATIONAL FUNCTIONS

DEF: A ***rational function*** is the quotient of two finite polynomials.

Example 7.4.3: $\dfrac{x + x^2}{1 - 3x + 3x^2 - x^3}$

Remark: Rational functions are among the most frequently encountered generating functions. One may think of a closed form as a way of "generating" the coefficients of a power series. In particular, division of polynomials generates the power series.

Example 7.4.4: Long division of polynomials

$$\begin{array}{r|l} & x + 4x^2 + 9x^3 + 16x^4 + \cdots \\ \hline 1 - 3x + 3x^2 - x^3 & x + x^2 \end{array}$$

Prop 7.4.1. *Every rational function is the OGF for a sequence.*

Pf: As illustrated by Example 7.4.4.

ENCODING A GIVEN SEQUENCE AS AN OGF

Several principles are valuable in constructing an OGF for a given sequence. Proofs are omitted.

Prop 7.4.2. *Two generating functions*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

have as their sum the generating function

$$(f+g)(x) = f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

Example 7.4.5: The Hanoi OGF

$$0 + 1x + 3x^2 + 7x^3 + 15x^4 + \cdots$$

is the sum of the OGF's

$$\frac{-1}{1-x} = \sum_{k=0}^{\infty} (-1) x^k \quad \text{and} \quad \frac{1}{1-2x} = \sum_{k=0}^{\infty} 2^k x^k$$

so its generating function is

$$\frac{-1}{1-x} + \frac{1}{1-2x} = \frac{x}{(1-x)(1-2x)}$$

Prop 7.4.3. *Two generating functions*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

have as their product the generating function

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

DEF: The ***convolution*** of the sequences $\langle a_j \rangle$ and $\langle b_k \rangle$ is the sequence

$$\left\langle \sum_{j=0}^{k} a_j b_{k-j} \right\rangle$$

Example 7.4.6: The sequence $1, 2, 3, 4 \ldots$ is the convolution of the sequence $1, 1, 1, \ldots$ with itself. Thus, its OGF is

$$\frac{1}{(1-x)^2}$$

Prop 7.4.4. *Multiplication of a OGF by x shifts all the coefficients one position to the right.* ♢

Example 7.4.7: The OGF for $0, 1, 2, 3, 4 \ldots$ is

$$\frac{x}{(1-x)^2}$$

With the theoretical tools now in hand, and with some ingenuity, we can construct many useful gen fcns.

Example 7.4.8: Construct a gen fcn for the sequence of squares:

$$0,\ 1,\ 4,\ 9,\ 16,\ 25,\ \ldots$$

Idea: $1 + 2 + \cdots + n \ = \ \dfrac{n^2 + n}{2}$

By Example 7.4.7,

$$\frac{2x}{(1-x)^3} \quad \text{generates} \quad n^2 + n$$

and

$$\frac{x}{(1-x)^2} \quad \text{generates} \quad n$$

Thus,

$$\frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2} \ = \ \frac{x^2 + x}{(1-x)^3}$$

is a gen fcn for n^2.

EXTENDED BINOMIAL THEOREM

Review: For any real number u and integer k, we define the ***falling power***

$$u^{\underline{k}} = u(u-1)\cdots(u-k+1)$$

and we define the ***extended binomial coeff***

$$\binom{u}{k} = \frac{u^{\underline{k}}}{k!}$$

Extended Binomial Theorem. *For every x such that $|x| < 1$ and for every $u \in \mathcal{R}$,*

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Pf: This is an immediate application of Maclaurin's theorem. $\diamondsuit$

Cor 7.4.5. *The OGF for the sequence*

$$\left\langle \frac{n^{\underline{r}}}{r!} \right\rangle$$

is

$$x^r(1-x)^{-r-1} \;=\; \frac{x^r}{(1-x)^{r+1}}$$

Example 7.4.9: OGF for the sequence $\langle n^3 \rangle$.

Step 1. $n^3 \;=\; n^{\underline{3}} + 3n^{\underline{2}} + n^{\underline{1}}$.

Step 2. $\langle n^3 \rangle \;=\; 6\left\langle \frac{n^{\underline{3}}}{3!} \right\rangle + 6\left\langle \frac{n^{\underline{2}}}{2!} \right\rangle + 1\left\langle \frac{n^{\underline{1}}}{1!} \right\rangle$

By Cor 7.4.5, the OGF for $\langle n^3 \rangle$ is

$$\frac{6x^3}{(1-x)^4} + \frac{6x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \;=\; \frac{x+4x^2+x^3}{(1-x)^4}$$

USING GEN FCNS TO SOLVE RR's

Using generating functions provides a single best method to solve RR's that avoids the need to memorize an endless bag of tricks.

Example 7.2.1, continued: We again solve the recurrence system

$$a_0 = 0 \qquad a_n = a_{n-1} + 2n - 1$$

this time, with generating functions.

$$a_n x^n = a_{n-1}x^n + 2nx^n + (-1)x^n \quad \text{mult by } x^n$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 2nx^n + \sum_{n=1}^{\infty}(-1)x^n$$

$$A(x) - a_0 = x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + \sum_{n=1}^{\infty} 2nx^n + \sum_{n=1}^{\infty}(-1)x^n$$

$$A(x) - 0 = x\sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} 2nx^n + \sum_{n=1}^{\infty}(-1)x^n$$

$$A(x) = xA(x) + \sum_{n=1}^{\infty} 2nx^n + \sum_{n=1}^{\infty}(-1)x^n$$

$$\begin{aligned}
A(x) &= xA(x) + \sum_{n=1}^{\infty} 2nx^n + \sum_{n=1}^{\infty}(-1)x^n \\
A(x) - xA(x) &= 2\sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty} x^n \\
A(x) - xA(x) &= \frac{2x}{(1-x)^2} - \frac{x}{1-x} \\
A(x)(1-x) &= \frac{x^2+x}{(1-x)^2} \\
A(x) &= \frac{x^2+x}{(1-x)^3}
\end{aligned}$$

Using Example 6.4.8, we recognize this as the OGF for the sequence of squares.

This method would work on Fibonacci or Catalan, but let's do some easy examples.

Example 7.2.6, continued: deg 2 recurrence.

$$a_0 = 1 \qquad a_1 = 4$$

$$a_n = 5a_{n-1} - 6a_{n-2}$$

recall problem solution: $a_n = 2 \cdot 3^n - 2^n$

This time, with generating functions.

$$a_n x^n = 5a_{n-1}x^n - 6a_{n-2}x^n \quad \text{mult by } x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 5\sum_{n=2}^{\infty} a_{n-1}x^n - 6\sum_{n=2}^{\infty} a_{n-2}x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 5x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 6x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n x^n = 5x\sum_{n=1}^{\infty} a_n x^n - 6x^2\sum_{n=0}^{\infty} a_n x^n$$

$$A(x) - 1 - 4x = 5x[A(x) - 1] - 6x^2 A(x)$$

$$A(x) - 1 - 4x = 5xA(x) - 5x - 6x^2 A(x)$$

$$\Rightarrow \quad A(x)[1 - 5x + 6x^2] = 1 - x$$

and therefore,

$$A(x) = \frac{1-x}{(1-2x)(1-3x)} = \frac{2}{1-3x} - \frac{1}{1-2x}$$

Thus, $a_n = 2 \cdot 3^n - 2^n$.

PARTIAL FRACTIONS

How do we obtain this decomposition?

$$A(x) = \frac{1-x}{(1-2x)(1-3x)} = \frac{2}{1-3x} - \frac{1}{1-2x}$$

ANSWER: like this!

$$\begin{aligned}\frac{1-x}{(1-2x)(1-3x)} &= \frac{b}{1-3x} + \frac{c}{1-2x} \\ &= \frac{b(1-2x)+c(1-3x)}{(1-2x)(1-3x)} \\ &= \frac{(b+c)-(2b+3c)x}{(1-2x)(1-3x)}\end{aligned}$$

Solving the simultaneous equations

$$\begin{aligned} b+c &= 1 \\ 2b+3c &= 1 \end{aligned}$$

yields the solution $b = 2 \qquad c = -1.$

Remark: These straighforward steps replace the bag-of-tricks for nonhomogeneity and repeated roots.

Example 7.2.8, continued: Hanoi again

$$\begin{aligned} h_0 &= 0 \\ h_n &= 2h_{n-1} + 1 \end{aligned}$$

recall problem solution: $h_n = 2^n - 1$.

This time, by generating functions.

$$\begin{aligned}
h_n x^n &= 2h_{n-1}x^n + x^n \quad \text{mult by } x^n \\
\sum_{n=1}^{\infty} h_n x^n &= 2\sum_{n=1}^{\infty} h_{n-1}x^n + \sum_{n=1}^{\infty} x^n \\
\sum_{n=1}^{\infty} h_n x^n &= 2x\sum_{n=1}^{\infty} h_{n-1}x^{n-1} + \sum_{n=1}^{\infty} x^n \\
H(x) - 0 &= 2xH(x) + \frac{x}{1-x} \\
H(x)[1-2x] &= \frac{x}{1-x} \quad \text{and therefore} \\
H(x) &= \frac{x}{(1-x)(1-2x)} = \frac{1}{1-2x} - \frac{1}{1-x}
\end{aligned}$$

Thus, $h_n = 2^n - 1$.

Example 7.2.11, continued: recurrence sys

$$a_0 = -2 \qquad a_1 = 2$$
$$a_n = 4a_{n-1} - 4a_{n-2}$$

recall problem solution: $a_n = (-2) \cdot 2^n + 3 \cdot n2^n$
This time, with generating functions.

$$a_n x^n = 4a_{n-1}x^n - 4a_{n-2}x^n \quad \text{mult by } x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 4\sum_{n=2}^{\infty} a_{n-1}x^n - 4\sum_{n=2}^{\infty} a_{n-2}x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 4x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n x^n = 4x\sum_{n=1}^{\infty} a_n x^n - 4x^2\sum_{n=0}^{\infty} a_n x^n$$

$$A(x) + 2 - 2x = 4x[A(x) + 2] - 4x^2 A(x)$$
$$A(x) + 2 - 2x = 4xA(x) + 8x - 4x^2 A(x)$$

$A(x)[1 - 4x + 4x^2] = 10x - 2$, and therefore,

$$A(x) = \frac{10x - 2}{(1-2x)^2} = \frac{6x}{(1-2x)^2} - \frac{2}{1-2x}$$

Thus, $a_n = 3n \cdot 2^n - 2 \cdot 2^n$.