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Chapter 2

Sets, Fcns, Seqs, Sums

2.1 SETS

DEF: A ***set*** is a collection of objects. The objects are called ***elements*** or ***members*** of the set.

NOTATION: : $x \in S$

Example 2.1.1:

- $2 \in \{5, -7, \pi, \text{"algebra"}, 2, 2.718\}$
- $8 \notin \{p : p \text{ is a prime number}\}$

SOME STANDARD SETS of NUMBERS

$\mathbb{N}$ = the set of all non-negative integers (the "natural numbers")

$\mathbb{Z}$ = the set of all integers

$\mathbb{Z}^+$ = the set of all positive integers

$$\mathbb{Q} = \text{the rationals} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

$\mathbb{R}$ = the real numbers

$\mathbb{C}$ = the complex numbers

ROSTERS for SETS

DEF: A ***roster*** specifies a finite set by enclosing in braces a list of representations of its elements. Repetitions and orderings are irrelevant to content.

Example 2.1.2: a roster

$$\{5, -7, \pi, \text{“algebra”}, 2, 2.718\}$$

Example 2.1.3: identical sets

$$\{1, 2, 3, 1, 1, 3\} = \{1, 2, 3\} = \{3, 1, 2\}$$

DEF: The ***empty set*** is the set $\{\ \}$ that has no elements. NOTATION: $\emptyset$.

Remark: In mathematics, there is only one empty set. However, a computer programming language may have a different empty set for every datatype.

Example 2.1.4: The empty set of character strings is equal to the empty set of lions.

DEF: A ***singleton set*** is a set with one element.

Example 2.1.5: $\{x\}$ is a singleton set.

SPECIFICATION by PREDICATES

A predicate over a well-defined set can specify any subcollection within that set. (Warning: This "set-builder" method can lead to non-sets.)

Example 2.1.6: $\{x \in \mathcal{Z} : P(x)\}$ where $P(x)$ is TRUE if x is prime.

Example 2.1.7: $\{(x, y) : x, y \in \mathcal{R} \wedge x^2 + y^2 = 1\}$

OTHER WAYS to SPECIFY SETS

(1) By prose. (can also lead to non-sets).

Example 2.1.8: The set of all palindromes.

(2) By operations on other sets.
Examples soon.

(3) By recursive construction.
Examples in §4.3.

SETS as ELEMENTS of SETS

Object x is not the same as the singleton set $\{x\}$. Moreover, $\{x\} \neq \{\{x\}\}$.

Analogy: Iterative pointers to a computer object creates new objects.

$$x \neq \&x \neq \&\&x$$

Analogy: Iterative enquotation of a character string creates new objects.

$$\text{``\,``lion''\,''} \neq \text{``lion''} \neq \text{lion}$$

RELATIONS on SETS

DEF: Set X is a ***subset*** of set Y if every element of X is also an element of Y. NOTATION: $X \subseteq Y$.

DEF: A subset X of a set Y is ***proper*** if Y has at least one element that is not in X.

DEF: Sets X and Y are ***equal*** if each set is a subset of the other. NOTATION: $X = Y$.

Example 2.1.9:

(1) $\emptyset$ is a proper subset of every set except itself.

(2) The integers are a subset of the real numbers.

DISAMBIGUATION: In computer languages, the integers and the reals are usually distinct ***datatypes***. Integer-type values are represented differently from real-type whole numbers.

Remark: Whereas mathematics deals with objects, computation science deals with their representations.

POWER SET

DEF: The ***power set*** of a set S is the set of all subsets of S. NOTATION: 2^S or $P(S)$.

Example 2.1.10:

- $P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
- $P(\emptyset) = \{\emptyset\}$.
- $P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$.

Proposition 2.1.1. *If set S has n elements, then the power set $P(S)$ has 2^n elements.* $\diamondsuit$

CARTESIAN PRODUCT

DEF: The ***cartesian product*** of sets A and B is the set

$$\{(a,b) \mid a \in A \wedge b \in B\}$$

NOTATION: $A \times B$.

Example 2.1.11: $A = \{1,2\}$ $B = \{a,b,c\}$
Then

$$A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$$

Proposition 2.1.2. *The cartesian product $A \times B$ is empty iff either A or B is empty.* $\diamondsuit$

2.2 SET OPERATIONS

Geometric figures were used by John Venn (1834-1923) to illustrate the effect of various operations on sets called the ***universal set*** or the ***domain of discourse***.

Remark: Not all set operations can be represented by Venn diagrams.

VENN DIAGRAMS

DEF: In a generic ***Venn diagram*** for a subset S of a fixed universal set U, the universal set is represented by a rectangular region in the plane and the set S is represented by a subregion.

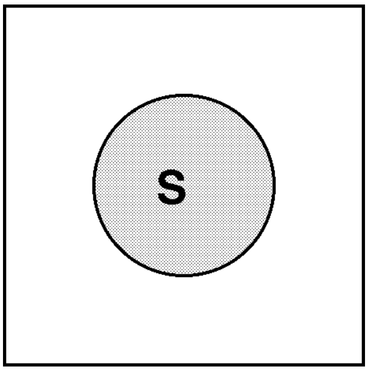


Fig 2.2.1 Set S is shaded.

DYADIC SET OPERATIONS

DEF: The ***union*** of sets S and T is the set containing every object that is either in S or in T. NOTATION: $S \cup T$.

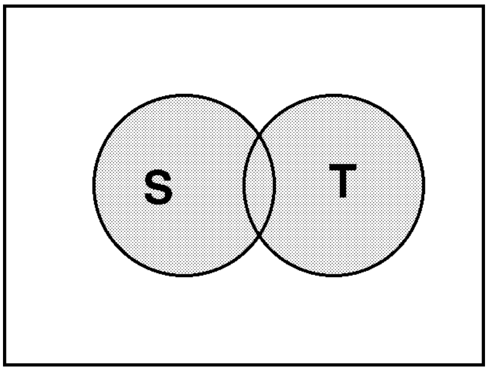


Fig 2.2.2 Union $S \cup T$ is shaded.

DEF: The ***intersection*** of sets S and T is the set containing every object that is in both S and T. NOTATION: $S \cap T$.

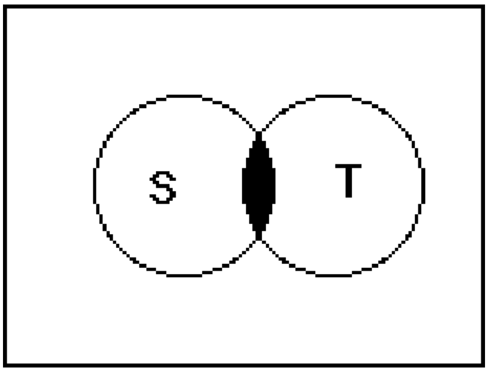


Fig 2.2.3 Intersection $S \cap T$ is shaded.

DEF: The ***difference*** of sets S and T is the set containing every object that is in S but not in T.

NOTATION: $S - T$.

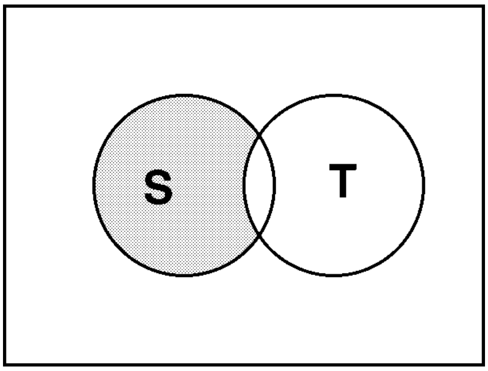


Fig 2.2.4 Difference $S - T$ is shaded.

Example 2.2.1:

$S = \{1, 3, 5, 7, 9\} \quad T = \{2, 3, 5, 7\}$. Then

$S \cup T = \{1, 2, 3, 5, 7, 9\}$.

$S \cap T = \{3, 5, 7\}$.

$S - T = \{1, 9\}$.

Example 2.2.2: Cartesian product (which is dyadic) is not representable by a Venn diagram.

MONADIC SET OPERATIONS

DEF: The ***complement*** of a set S is the set $U - S$, where U is the universal set. NOTATION: $\overline{S}$.

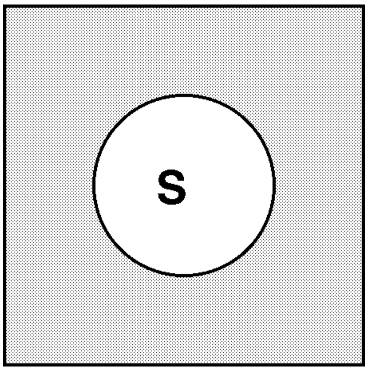


Fig 2.2.5 Complement $\overline{S}$ is shaded.

Example 2.2.3: These monadic operations are not readily representable by Venn diagrams.

$S \to \{S\}$ (***enbracement***)

$S \to P(S)$ (***empowerment***)

TERMINOLOGY NOTE: These two (original) names have excellent mnemonicity. The mildly frivolous character may deter their widespread adoption.

SET IDENTITIES

Various set equivalences have earned the honorific appelation ***i***dentity. Many of them are analogous to the logical equivalences of §1.2.

Example 2.2.4: The ***Double Negation Law***

$$\neg\neg p \Leftrightarrow p$$

has the following set-theoretic analogy:

DEF: ***Double Complementation Law:***

$$\overline{\overline{S}} = S$$

Example 2.2.5: The tautology $p \vee \neg p$ is called the ***Law of the Excluded Middle***. It converts to the equivalence

$$p \vee \neg p \Leftrightarrow T$$

which has the following set-theoretic analogy:

$$S \cup \overline{S} = U$$

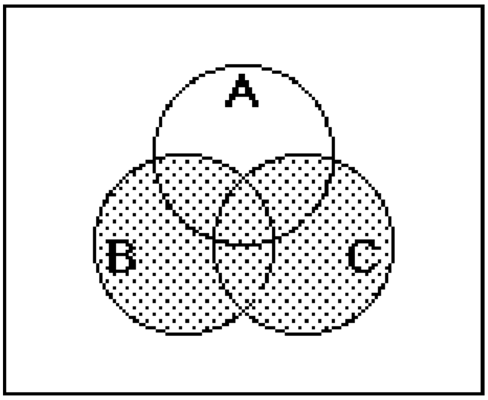
AVOIDING BOREDOM

Example 2.2.6: Table 1 of §2.2 (de Morgan, associativity, etc.) is good for self-study, but not for exhaustive classroom presentation.

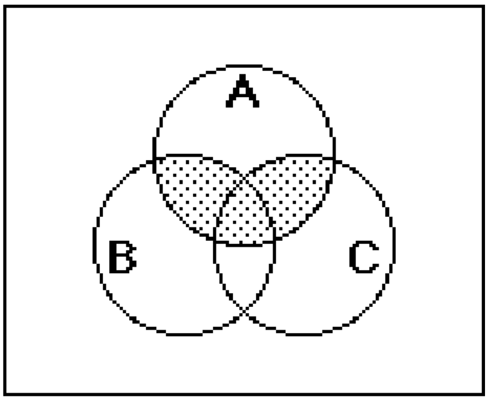
CONFIRMING IDENTITIES with VENN DIAGRAMS

Example 2.2.7: $\cap$ distributes over $\cup$.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



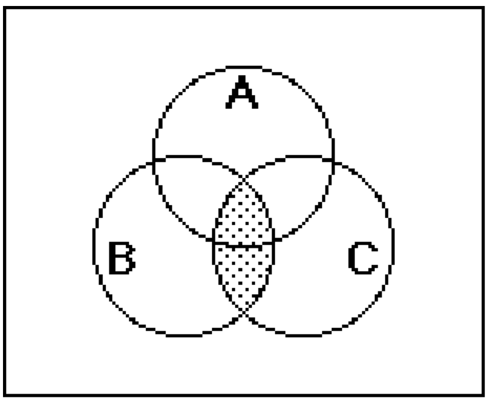
B ∪ C



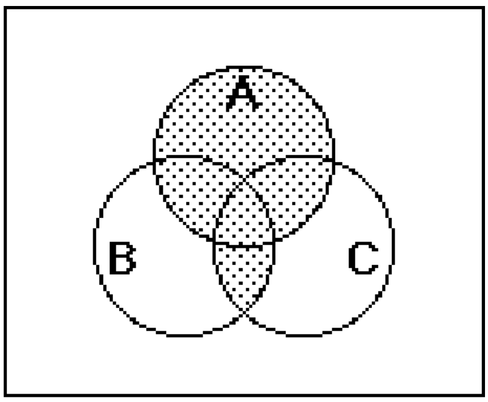
A ∩ (B ∪ C)

Example 2.2.8: $\cup$ distributes over $\cap$.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$



B ∩ C



A ∪ (B ∩ C)

2.3 FUNCTIONS

DEF: Let A and B be sets. A ***function*** $f : A \to B$ (often abbreviated as f) is a rule that assigns to each element $a \in A$ exactly one element $f(a) \in B$, called the ***value of the function*** f at a.

- We also say that $f : A \to B$ is a ***mapping*** from ***domain*** A to ***codomain*** B.
- $f(a)$ is called the ***image set of the element*** a, and the element a is called a ***preimage*** of $f(a)$.
- The set $\{a \mid f(a) = b\}$ is called the ***preimage set*** of b. NOTATION: $f^{-1}(b)$.

DEF: The set $\{b \in B \mid (\exists a \in A)[f(a) = b]\}$ is called the ***image of the function*** $f : A \to B$.

DEF: The word ***range*** is commonly used to mean the image set.

DEF: A function is called ***discrete*** if its domain and codomain are both finite or countable (indexed by $\mathbb{Z}$).

Example 2.3.1: Some functions from $\mathbb{R}$ to $\mathbb{Z}$.

(1) ***floor*** $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$ image $= \mathbb{Z}$

(2) ***ceiling*** $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$ im $= \mathbb{Z}$

(3) ***sign*** $\sigma(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & x > 0 \end{cases}$

image$(\sigma) = \{-1, 0, +1\}$

Example 2.3.2: Seq of functions from $\mathbb{R}$ to $\mathbb{R}$.

falling powers $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$.

$7^{\underline{3}} = 7{\cdot}6{\cdot}5 = 210 \quad \left(\frac{3}{2}\right)^{\underline{3}} = \frac{3}{2}\cdot\frac{1}{2}\cdot\left(\frac{-1}{2}\right) = \frac{-3}{8}$

Example 2.3.3: Functions in computation.

(1) ***C compiler*** maps the set of ASCII strings to the boolean set.

(2) The ***halting function*** maps the set of C programs to the boolean set, assigns TRUE iff this program will always halt eventually, no matter what input is supplied at run time.

Theorem 2.3.1. *The halting function cannot be represented by a C program.* $\diamondsuit$ *(CS W3261)*

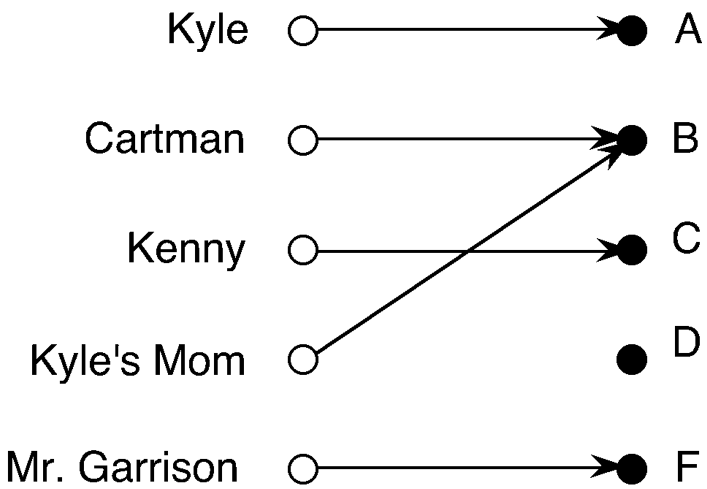
REPRESENTATION of DISCRETE FUNCTIONS

DEF: The $n \times 2$ ***array representation*** of a discrete function is a table with two columns. The left column contains every element of the domain. The second entry in each row is the image of the first entry.

DEF: The ***(full) digraphic representation*** of a discrete function is a diagram with two columns of dots. The left column contains a dot for every element of the domain, and the right entry contains a dot for every element of the codomain. From each domain dot an arrow is drawn to the codomain dot representing its image.

Example 2.3.4: Representing a function.

student	grade
Kyle	A
Cartman	B
Kenny	C
Kyle's Mom	B
Mr. Garrison	F



ONE-TO-ONE and ONTO FUNCTIONS

DEF: A function $f : A \to B$ is ***one-to-one*** if for every $b \in B$, there is at most one $a \in A$ such that $f(a) = b$.

Prop 2.3.2. *A discrete function is one-to-one iff in its digraphic representation, no codomain dot is at the head of more than one arrow.* $\diamondsuit$

DEF: A function $f : A \to B$ is ***onto*** if

> for every $b \in B$, there is at least one $a \in A$ such that $f(a) = b$.

Prop 2.3.3. *A discrete function is onto iff in its digraphic representation, every codomain dot is at the head of at least one arrow.* $\diamondsuit$

Example 2.3.5: The grading function of Example 2.3.4 is neither one-to-one or onto.

BIJECTIONS

DEF: A ***bijection*** is a function that is one-to-one and onto.

DEF: Let $f : A \to B$ be a bijection. The ***inverse function***

$$f^{-1} : B \to A$$

is the rule that assigns to each $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

Example 2.3.6: The function

$$\{1 \mapsto b,\ 2 \mapsto c,\ 3 \mapsto a\}$$

is a bijection. Its inverse is the function

$$\{a \mapsto 3,\ b \mapsto 1,\ c \mapsto 2\}$$

DEF: A ***permutation*** is a bijection whose domain and codomain are the same set.

Example 2.3.7: The function

$$\{1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 1\}$$

is a permutation. Its inverse is the permutation

$$\{1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2\}$$

2.4 SEQUENCES AND SUMS

DEF: A ***sequence in a set*** A is a function f from a subset of the integers (usually $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$) to A. The values of a sequence are also called ***terms*** or ***entries***.

NOTATION: The value $f(n)$ is usually denoted a_n. A sequence is often written

$$a_0, \quad a_1, \quad a_2, \quad \ldots$$

Example 2.4.1: Two sequences.

$$a_n = \frac{1}{n} \qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

$$b_n = (-1)^n \quad 1, -1, 1, -1, \ldots$$

Example 2.4.2: Five ubiquitous sequences.

$$\begin{array}{rl} n^2 & 0, 1, 4, 9, 16, 25, 36, 49, \ldots \\ n^3 & 0, 1, 8, 27, 64, 125, 216, 343, \ldots \\ 2^n & 1, 2, 4, 8, 16, 32, 64, 128, \ldots \\ 3^n & 1, 3, 9, 27, 81, 243, 729, 2187, \ldots \\ n! & 1, 1, 2, 6, 24, 120, 720, 5040, \ldots \end{array}$$

STRINGS

DEF: A set of characters is called an ***alphabet***.

Example 2.4.3: Some common alphabets:

- $\{0, 1\}$ the binary alphabet
- $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ the decimal digits
- $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$ the hexadecimal digits
- $\{A, B, C, D, \ldots, X, Y, Z\}$ English uppercase
- ASCII

DEF: A ***string*** is a sequence in an alphabet.

NOTATION: Usually a string is written without commas, so that consecutive characters are juxtaposed.

Example 2.4.4: If

$$f(0) = M,\ f(1) = A,\ f(2) = T, \text{ and } f(3) = H$$

then write

$$\text{“}MATH''$$

SPECIFYING a RULE

Problem: Given some initial terms

$$a_0, a_1, \ldots, a_k$$

of a sequence, try to construct a rule that is consistent with those initial terms.

Approaches: There are two standard kinds of rule for calculating a generic term a_n.

DEF: A ***recursion*** for a_n is a function whose arguments are earlier terms in the sequence.

DEF: A ***closed form*** for a_n is a formula whose argument is the subscript n.

Example 2.4.5: $1, 3, 5, 7, 9, 11, \ldots$

recursion: $a_0 = 1; \quad a_n = a_{n-1} + 2$ for $n \geq 1$

closed form: $a_n = 2n + 1$

The differences between consecutive terms often suggest a recursion. Finding a recursion is usually easier than finding a closed formula.

Example 2.4.6: $1, 3, 7, 13, 21, 31, 43, \ldots$

recursion: $b_0 = 1; \quad b_n = b_{n-1} + 2n$ for $n \geq 1$

closed form: $b_n = n^2 + n + 1$

Sometimes, constructing a closed formula is much harder than constructing a recursion.

Example 2.4.7: $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

recursion: $c_0 = 1, c_1 = 1;$

$$c_n = c_{n-1} + c_{n-2} \text{ for } n \geq 1$$

closed form: $c_n = \frac{1}{\sqrt{5}}\left[G^{n+1} - g^{n+1}\right]$

where $G = \frac{1+\sqrt{5}}{2}$ and $g = \frac{1-\sqrt{5}}{2}$

INFERRING a RULE

The ESSENCE of science is inferring rules from partial data.

Example 2.4.8: Sit under apple tree. Infer gravity.

Example 2.4.9: Watch starlight move 0.15 arc-seconds in total eclipse. Infer relativity.

Example 2.4.10: Observe biological species. Infer DNA.

Important life skill: Given a difficult general problem, start with special cases you can solve.

Example 2.4.11: Find a recursion and a closed form for the arithmetic progression:

$$c, c+d, c+2d, c+3d, \ldots$$

recursion: $a_0 = c; \quad a_n = a_{n-1} + d$

closed form: $a_n = c + nd$.

Q: How would you decide that a given sequence is an arithmetic progression?

A: Calculate differences betw consec terms.

DEF: The ***difference sequence*** for a sequence a_n is the sequence $a'_n = a_n - a_{n-1}$ for $n \geq 1$.

Example 3.2.5 redux:

$$\begin{array}{llllllll} a_n: & 1 & 3 & 5 & 7 & 9 & 11 \\ a'_n: & 2 & 2 & 2 & 2 & 2 & \end{array}$$

Analysis: Since a'_n is constant, the sequence is specified by this recursion:
$a_0 = 1; a_n = a_{n-1} + 2$ for $n \geq 1$.

Moreover, it has this closed form:

$$\begin{aligned} a_n &= a_0 + a'_1 + a'_2 + \cdots + a'_n \\ &= a_0 + 2 + 2 + \cdots + 2 \quad = 1 + 2n \end{aligned}$$

If you don't get a constant sequence on the first difference, then try reiterating.

Revisit Example 3.2.6: $1, 3, 7, 13, 21, 31, 43, \ldots$

$$\begin{array}{lccccccc} b_n: & 1 & 3 & 7 & 13 & 21 & 31 & 43 \\ b'_n: & 2 & 4 & 6 & 8 & 10 & 12 & \\ b''_n: & 2 & 2 & 2 & 2 & 2 & & \end{array}$$

Analysis: Since b''_n is constant, we have

$$b'_n = 2 + 2n$$

Therefore,

$$\begin{aligned} b_n &= b_0 + b'_1 + b'_2 + \cdots + b'_n \\ &= b_0 + 2\sum_{j=1}^{n} j = 1 + (n^2 + n) = n^2 + n + 1 \end{aligned}$$

Consolation Prize: Without knowing about finite sums, you can still extend the sequence:

$$\begin{array}{lcccccccc} b_n: & 1 & 3 & 7 & 13 & 21 & 31 & 43 & \underline{57} \\ b'_n: & 2 & 4 & 6 & 8 & 10 & 12 & \underline{14} & \\ b''_n: & 2 & 2 & 2 & 2 & 2 & \underline{2} & & \end{array}$$

SUMMATIONS

DEF: Let a_n be a sequence. Then the ***big-sigma*** notation

$$\sum_{j=m}^{n} a_j$$

means the sum

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

TERMINOLOGY: j is the ***index of summation***

TERMINOLOGY: m is the ***lower limit***

TERMINOLOGY: n is the ***upper limit***

TERMINOLOGY: a_j is the ***summand***

Theorem 2.4.1. *These formulas for summing falling powers are provable by induction (see §4.1):*

$$\sum_{j=1}^{n} j^{\underline{1}} = \frac{1}{2}(n+1)^{\underline{2}} \qquad \sum_{j=1}^{n} j^{\underline{2}} = \frac{1}{3}(n+1)^{\underline{3}}$$

$$\sum_{j=1}^{n} j^{\underline{3}} = \frac{1}{4}(n+1)^{\underline{4}} \qquad \sum_{j=1}^{n} j^{\underline{k}} = \frac{1}{k+1}(n+1)^{\underline{k+1}}$$

Example 2.4.12: True Love and Thm 2.4.1

On the j^{th} day ... True Love gave me

$$j + (j-1) + \cdots + 1 \;=\; \frac{(j+1)^{\underline{2}}}{2} \text{ gifts.}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{j=2}^{13} j^{\underline{2}} \;=\; \frac{1}{2}\left[2^{\underline{2}} + \cdots + 13^{\underline{2}}\right] \\
&= \frac{1}{2}\left[2 + 6 + \cdots + 78\right] \;=\; 364 \quad \text{slow} \\
&= \frac{1}{2}\cdot\frac{14^{\underline{3}}}{3} \;=\; 364 \quad \text{fast}
\end{aligned}$$

Cor 2.4.2. *High-powered look-ahead to formulas for summing* $j^k : j = 0, 1, ..., n$.

$$\sum_{j=1}^{n} j^2 \;=\; \sum_{j=1}^{n}\left(j^{\underline{2}} + j^{\underline{1}}\right) \;=\; \frac{1}{3}(n+1)^{\underline{3}} + \frac{1}{2}(n+1)^{\underline{2}}$$

$$\sum_{j=1}^{n} j^3 \;=\; \sum_{j=1}^{n}\left(j^{\underline{3}} + 3j^{\underline{2}} + j^{\underline{1}}\right) \;=\; \cdots$$

POTLATCH RULES for CARDINALITY

DEF: ***nondominating cardinality:***
Let A and B be sets. Then $|A| \leq |B|$ means that $\exists$ one-to-one function $f : A \to B$.

DEF: Set A and B have ***equal cardinality***, and we write $|A| = |B|$, if $\exists$ bijection $f : A \to B$, which obviously implies that $|A| \leq |B|$ and $|B| \leq |A|$.

DEF: ***strictly dominating cardinality:***
Let A and B be sets. Then $|A| < |B|$ means that $|A| \leq |B|$ and $|A| \neq |B|$.

DEF: The ***cardinality*** of a set A is

$$|A| \;=\; \begin{cases} n & \text{if } |A| = |\{1, 2, \ldots, n\}| \\ 0 & \text{if } A = \emptyset \end{cases}$$

Such cardinalities are called ***finite***.

DEF: The ***cardinality*** of $\mathbb{N}$ is ω ("omega"), or alternatively, $\aleph_0$ ("aleph null").

DEF: A set is ***countable*** if it is finite or ω.

Remark: $\aleph_0$ is the smallest infinite cardinality. The set $\mathbb{R}$ has cardinality $\aleph_1$ ("aleph one"), which is larger than $\aleph_0$, for reasons to be given.

INFINITE CARDINALITIES

Proposition 2.4.3. *There are as many even non-negative numbers as non-negative numbers.*

Pf: $f(2n) = n$ is a bijection. $\diamondsuit$

Theorem 2.4.4. *There are as many positive integers as rational fractions.*

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Pf: $f\left(\frac{p}{q}\right) = \frac{(p+q-1)(p+q-2)}{2} + p$ $\diamondsuit$

Example 2.4.13: $f\left(\frac{2}{3}\right) = \frac{4 \cdot 3}{2} + 2 = 8$

Thm 2.4.5. *(G. Cantor) There are more positive real numbers than positive integers.*

Semi-proof: A putative bijection $[0, 1] \to \mathbb{Z}^+$ would induce a sequence x_j such that $\{x_j\} = [0, 1]$. Suppose we write each real number as an infinite decimal fraction.

$$x_1 = .\underline{8}841752032669031 \ldots \mapsto 1$$
$$x_2 = .1\underline{4}15926531424450 \ldots \mapsto 2$$
$$x_3 = .32\underline{0}2313932614203 \ldots \mapsto 3$$
$$x_4 = .167\underline{9}888138381728 \ldots \mapsto 4$$
$$x_5 = .0452\underline{9}98136712310 \ldots \mapsto 5$$
$$\vdots$$

The j^{th} digit of x_j is underscored. Consider the number

$$.73988 \ldots$$

whose j^{th} decimal digit differs (by $1 \mod 10$) from the j^{th} digit of x_j, which implies that

$$.73988 \ldots \not\mapsto \; j \qquad (\forall j \in \mathbb{Z}+)$$

Accordingly, this putative bijection is not even a function, since it fails to assign an element of the codomain $\mathbb{Z}^+$ to some number of its domain. $\Diamond$