The Curry-Howard Correspondence

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Much of this was taken from

Dan Grossman, CS152, Lecture 12, Spring 2011

`https://homes.cs.washington.edu/~djg/2011sp/`

Logic

Caltech graduates are nerds.

Stephen is a Caltech graduate.

Is Stephen a nerd?

Propositional Logic

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p	Stephen went to Caltech	Proposition
q	Stephen is a nerd	Proposition
$p \longrightarrow q$	If Stephen went to Caltech, he is a nerd	Implication

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"method to put" in Latin

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Propositional Logic

b base propositions

$p ::= b \mid p \longrightarrow p$

$\Gamma ::= \varnothing \mid \Gamma, p$

$$\frac{p \in \Gamma}{\Gamma \vdash p}$$

$$\frac{\Gamma \vdash p_1 \longrightarrow p_2 \quad \Gamma \vdash p_1}{\Gamma \vdash p_2}\ \textit{Modus Ponens}$$

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Simply-Typed Lambda Calculus

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b base types

Propositional Logic | Simply-Typed Lambda Calculus

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Propositions are Types

b base types

$\tau ::= b \mid \tau \longrightarrow \tau$

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b base types

Propositions are Types $\quad \tau ::= b \mid \tau \longrightarrow \tau$

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Proofs are Programs $\quad e ::= x \mid e\ e \mid \lambda x\ .\ e$

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$$\frac{p \in \Gamma}{\Gamma \vdash p}$$

$$\frac{\tau \in \Gamma}{\Gamma \vdash \quad \tau}\ \text{t-var}$$

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Propositions are Types

Proofs are Programs

Simply-Typed Lambda Calculus

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$\Gamma ::= \emptyset \mid \Gamma, x : \tau$

$e ::= x \mid e\ e \mid \lambda x\ .\ e$

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}\ \text{t-var}$$

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Propositional Logic		Simply-Typed Lambda Calculus
b base propositions		b base types
$p ::= b \mid p \longrightarrow p$	Propositions are Types	$\tau ::= b \mid \tau \longrightarrow \tau$
$\Gamma ::= \varnothing \mid \Gamma, p$		$\Gamma ::= \varnothing \mid \Gamma, x : \tau$
	Proofs are Programs	$e ::= x \mid e\ e \mid \lambda x . e$

$$\frac{p \in \Gamma}{\Gamma \vdash p} \qquad\qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}\ \text{t-var}$$

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Simply-Typed Lambda Calculus

b base types

Propositions are Types: $\tau ::= b \mid \tau \longrightarrow \tau$

$\Gamma ::= \varnothing \mid \Gamma, x : \tau$

Proofs are Programs: $e ::= x \mid e\ e \mid \lambda x\ .\ e$

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}\ \text{t-var}$$

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$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash (\lambda x\ .\ e) : \tau_1 \longrightarrow \tau_2}\ \text{t-abs}$$

Haskell as a Proof Assistant

```
$ ghci
> data StephenTecher = StephenIsATecher    -- Propositions
> data StephenNerd   = StephenIsANerd
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> s = StephenIsATecher                      -- Assumption 1:
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s :: StephenTecher                          --   (See my diploma)
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s :: StephenTecher                         --   (See my diploma)

> isnerd StephenIsATecher = StephenIsANerd -- Assumption 2:
> :t isnerd                                -- If he's a techer,
isnerd :: StephenTecher -> StephenNerd     --   he's a nerd
```

Haskell as a Proof Assistant

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$ ghci
> data StephenTecher = StephenIsATecher    -- Propositions
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> isnerd StephenIsATecher = StephenIsANerd -- Assumption 2:
> :t isnerd                                -- If he's a techer,
isnerd :: StephenTecher -> StephenNerd     --   he's a nerd

> duh = isnerd s :: StephenNerd            -- Check the proof
> :t duh                                   -- No errors so yes,
duh :: StephenNerd                         --   Stephen is a nerd
```

Coq: An Actual Proof Assistant

```
Parameters (StephenTecher : Prop)                              (* Propositions *)
           (StephenNerd   : Prop).
```

Coq: An Actual Proof Assistant

```
Parameters (StephenTecher : Prop)                   (* Propositions *)
           (StephenNerd   : Prop).

Hypotheses (stephen : StephenTecher)                (* Assumptions *)
           (isnerd  : StephenTecher -> StephenNerd).
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Coq: An Actual Proof Assistant

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Parameters (StephenTecher : Prop)                    (* Propositions *)
           (StephenNerd   : Prop).

Hypotheses (stephen : StephenTecher)                 (* Assumptions *)
           (isnerd  : StephenTecher -> StephenNerd).

Definition duh : StephenNerd := isnerd stephen.      (* A proof *)

Print duh.                                           (* print the proof *)

duh = isnerd stephen
  : StephenNerd
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Parameters (StephenTecher : Prop)                  (* Propositions *)
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Definition duh : StephenNerd := isnerd stephen.    (* A proof *)

Print duh.                                         (* print the proof *)

duh = isnerd stephen
  : StephenNerd

Theorem duh': StephenNerd.                        (* More traditional *)
Proof.
  exact (isnerd stephen).
Qed.
```

More Propositional Logic

$$p ::= b \mid p \longrightarrow p$$

STLC + pairs + sums

$$\tau ::= b \mid \tau \longrightarrow \tau$$

$$e ::= x \mid \lambda x \;.\; e$$

More Propositional Logic

$$p ::= b \mid p \longrightarrow p \mid p \wedge p$$

Logical AND

STLC + pairs + sums

$$\tau ::= b \mid \tau \longrightarrow \tau$$

$$e ::= x \mid \lambda x \,.\, e$$

More Propositional Logic

$$p ::= b \mid p \longrightarrow p \mid p \land p$$

$$\frac{\Gamma \vdash p_1 \qquad \Gamma \vdash p_2}{\Gamma \vdash p_1 \land p_2}\text{and-c}$$

STLC + pairs + sums

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$$\frac{\Gamma \vdash p_1 \wedge p_2}{\Gamma \vdash p_1}\text{and-d1} \qquad \frac{\Gamma \vdash p_1 \wedge p_2}{\Gamma \vdash p_2}\text{and-d2}$$

STLC + pairs + sums

$$\tau ::= b \mid \tau \longrightarrow \tau$$

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What type do we need for AND?

More Propositional Logic

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STLC + pairs + sums

$$\tau ::= b \mid \tau \longrightarrow \tau \mid \tau \times \tau$$

$$e ::= x \mid \lambda x \,.\, e$$

Product Type: Pairs/Tuples

In C, `struct { int fst; char snd; };`

In Haskell, `(Int, Char)`

In OCaml, `int * char`

More Propositional Logic

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$$\tau ::= b \mid \tau \longrightarrow \tau \mid \tau \times \tau$$

$$e ::= x \mid \lambda x \,.\, e \mid (e, e) \mid \text{fst } e \mid \text{snd } e$$

E.g., (1,2) constructs a pair

fst (1,2) = 1

snd (1,2) = 2

More Propositional Logic

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$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

Build a pair from expressions of type τ_1 and τ_2

More Propositional Logic

$$p ::= b \mid p \longrightarrow p \mid p \wedge p$$

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Take apart a pair; result depends on selected element

More Propositional Logic

$$p ::= b \mid p \longrightarrow p \mid p \wedge p \mid p \vee p$$

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$$\frac{\Gamma \vdash p_1}{\Gamma \vdash p_1 \vee p_2}\text{ or-c1} \qquad \frac{\Gamma \vdash p_2}{\Gamma \vdash p_1 \vee p_2}\text{ or-c2}$$

Logical OR

STLC + pairs + sums

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$$\frac{\Gamma \vdash p_1 \vee p_2 \qquad \begin{array}{c}\Gamma, p_1 \vdash p_3 \\ \Gamma, p_2 \vdash p_3\end{array}}{\Gamma \vdash p_3}\ \text{or-d}$$

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What type do we need for OR?

More Propositional Logic

$$p ::= b \mid p \longrightarrow p \mid p \wedge p \mid p \vee p$$

$$\frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1 \wedge p_2}\text{ and-c}$$

$$\frac{\Gamma \vdash p_1 \wedge p_2}{\Gamma \vdash p_1}\text{ and-d1} \qquad \frac{\Gamma \vdash p_1 \wedge p_2}{\Gamma \vdash p_2}\text{ and-d2}$$

$$\frac{\Gamma \vdash p_1}{\Gamma \vdash p_1 \vee p_2}\text{ or-c1} \qquad \frac{\Gamma \vdash p_2}{\Gamma \vdash p_1 \vee p_2}\text{ or-c2}$$

$$\frac{\Gamma \vdash p_1 \vee p_2 \quad \begin{array}{c}\Gamma, p_1 \vdash p_3 \\ \Gamma, p_2 \vdash p_3\end{array}}{\Gamma \vdash p_3}\text{ or-d}$$

STLC + pairs + sums

$$\tau ::= b \mid \tau \longrightarrow \tau \mid \tau \times \tau \mid \tau + \tau$$

$$e ::= x \mid \lambda x\ .\ e \mid (e, e) \mid \text{fst } e \mid \text{snd } e$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{fst } e : \tau_1} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{snd } e : \tau_2}$$

A sum or union type: can be either τ_1 or τ_2

C: `union { int l; char r; };`

Haskell: `data Sum a b = L a | R b`

Ocaml: `type ('a, 'b) sum = L of 'a | R of 'b`

More Propositional Logic

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$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{fst}\ e : \tau_1} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{snd}\ e : \tau_2}$$

“L e” and “R e” are constructors for the variants

“$e\ ?\ e\ :\ e$” is the destructor

“L $e\ ?\ e_2\ :\ e_3$” evaluates to $e_2[l := e]$

“R $e\ ?\ e_2\ :\ e_3$” evaluates to $e_3[r := e]$

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$$\frac{\Gamma \vdash p_1 \vee p_2 \qquad \begin{matrix}\Gamma, p_1 \vdash p_3 \\ \Gamma, p_2 \vdash p_3\end{matrix}}{\Gamma \vdash p_3}\ \text{or-d}$$

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$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{L}\ e : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{R}\ e : \tau_1 + \tau_2}$$

L's argument must be a τ_1

R's argument must be a τ_2

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“$e_1\ ?\ e_2 : e_3$” means

```
match e1 with
L l -> e2
R r -> e3
```

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$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \quad \begin{array}{c}\Gamma, l : \tau_1 \vdash e_2 : \tau_3 \\ \Gamma, r : \tau_2 \vdash e_3 : \tau_3\end{array}}{\Gamma \vdash (e_1\ ?\ e_2 : e_3) : \tau_3}$$