Simplicial Complex Sampling in Inference using Exact Sequences

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Abstract

A space of signals on a graph can be represented using an abstract mathematical construction called a *sheaf* – a general tool that allows the creation of *topological filters*. If only partial information about a graph signal is present – yielding a sampling problem analogous to the Shannon-Nyquist theorem – then an exact sequence of sheaves provides detailed information about how this information can be extrapolated. This general framework applies to many kinds of information integration problems and leads directly to computable invariants about graph signal processing filters.

The Shannon-Nyquist sampling theorem states that sampling a signal at twice its bandwidth is sufficient to reconstruct the signal. Its wide applicability leads to the question of whether there exist similar conditions for reconstructing other data from samples in more general settings. This article – summarizing [7, 6, 8] – explains that perfect reconstruction for sampling of local algebraic data on simplicial complexes can be addressed through the machinery exact sequences of cellular sheaves. As a demonstration of our technique, we examine conditions for perfect reconstruction of piecewise linear signals on graphs.

Sheaf theory has not been used in applications until fairly recently. The catalyst for new applications was the technical tool of *cellular sheaves*, developed in [9]. Since that time, an applied sheaf theory literature has emerged, for instance [2, 3, 1, 4, 5].

1 Cellular sheaves

A sheaf is a mathematical object that stores locally-defined data over a space. In order to formalize this concept, we need a concept of space that is convenient for computations. The most efficient such definition is that of a simplicial complex.

Definition 1. (see [9]) A *sheaf* F on an abstract simplicial complex X is a covariant functor from the face category of X to the category of vector spaces. Explicitly,

- for each element a of X, $F(a)$ is a vector space, called the *stalk at* a,
- for each inclusion of two faces $a \to b$ of X, $F(a \to b)$ is a linear function from $F(a) \to F(b)$ called a *restriction*, and
- for every composition of inclusions $a \to b \to c$, $F(b \to c) \circ F(a \to b) = F(a \to b \to c)$.

Definition 2. Suppose F is a sheaf on an abstract simplicial complex X and that $\mathcal{U}$ is a collection of faces of X. An assignment s which assigns an element of $F(u)$ to each face $u \in \mathcal{U}$ is called a *section* supported on $\mathcal{U}$ when for each inclusion $a \to b$ (in X) of objects in $\mathcal{U}$, $F(a \to b)s(a) = s(b)$. A *global section* is a section supported on X.

Example 3. Consider $Y \subseteq X$ a subset of the vertices of an abstract simplicial complex. The functor S which assigns a vector space V to vertices in Y and the trivial vector space to every other face is called a *V-sampling sheaf supported on Y*. To every inclusion between faces of different dimension, S will assign the zero function. For a finite abstract simplicial complex X, the space of global sections of a V-sampling sheaf supported on Y is isomorphic to $\bigoplus_{y \in Y} V$.

Definition 4. A *morphism* $f : F \to G$ of sheaves on an abstract simplicial complex X assigns a linear map $f_a : F(a) \to G(a)$ to each face a so that for every inclusion $a \to b$ in the face category of X, $f_b \circ F(a \to b) = G(a \to b) \circ f_a$.

Theorem 5. *[7] Every FIR LTI filter F arises as the composition of linear maps $F = \lambda_* \circ p_*^{-1} : \Gamma S_1 \to \Gamma S_3$ induced on global sections by a pair of sheaf morphisms*

$$S_1 \xleftarrow{p} S_2 \xrightarrow{\lambda} S_3.$$

1.1 Sheaf cohomology

Define the following formal *cochain* vector spaces $C^k(X; F) = \bigoplus_{a \text{ a } k\text{-face of } X} F(a)$. The *coboundary map* $d^k : C^k(X; F) \to C^{k+1}(X; F)$ takes an assignment s from the k faces to an assignment $d^k s$ whose value at a $k+1$ face b is

$$(d^k s)(b) = \sum_{a \text{ a } k\text{-face of} X} [b : a] F(a \to b) s(a)$$

where $[b : a]$ captures the sign of the relative orientation of faces a and b. It can be shown that $d^k \circ d^{k-1} = 0$, so that the image of d^{k-1} is a subspace of the kernel of d^k.

Definition 6. The k-th *sheaf cohomology* of F on an abstract simplicial complex X is

$$H^k(X; F) = \ker d^k / \text{image } d^{k-1}.$$

Observe that $H^0(X; F) = \ker d^0$ consists precisely of those assignments s which are global sections. Cohomology is also a functor: sheaf morphisms induce linear functions between cohomologies. This indicates that cohomology preserves and reflects the underlying relationships between sheaves.

2 The Nyquist criterion for sheaves

Suppose that F is a sheaf on an abstract simplicial complex X, and that S is a V-sampling sheaf on X supported on a closed subcomplex Y. A *sampling* of F is a morphism $s : F \to S$ that is surjective on every stalk. Given a sampling, we can construct the *ambiguity sheaf* A in which the stalk $A(a)$ for a face $a \in X$ is given by the kernel of the map $F(a) \to S(a)$. If $a \to b$ is an inclusion of faces in X, then $A(a \to b)$ is $F(a \to b)$ restricted to $A(a)$. This implies that

$$0 \to A \hookrightarrow F \xrightarrow{\ s\ } S \to 0$$

is an exact sequence, which induces the long exact sequence (via the Snake lemma)

$$0 \to H^0(X; A) \to H^0(X; F) \to H^0(X; S) \to H^1(X; A) \to$$

An immediate consequence is therefore

Theorem 7. *(Sheaf-theoretic Nyquist theorem, [6]) The global sections of F are identical with the global sections of S if and only if $H^k(X; A) = 0$ for $k = 0$ and 1.*

The cohomology space $H^0(X; A)$ characterizes the *ambiguity* in the sampling, while $H^1(X; A)$ characterizes its *redundancy*. Optimal sampling therefore consists of identifying minimal closed subcomplexes Y so the resulting ambiguity sheaf A has $H^0(X; A) = H^1(X; A) = 0$.

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