# Local Construction of Bounded-Degree Network Topologies Using Only Incidence Information

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## Abstract

We consider ad-hoc networks consisting of $n$ wireless nodes that are located on $\mathbb{R}^2$. Any two given nodes are called neighbors if they are located within a certain distance from one another. A given node can be directly connected to any one of its neighbors and picks its connections according to a unique topology control algorithm that is available at every node. Given that each node knows only the indices of its one- and two-hop neighbors, we identify an algorithm that preserves connectivity and can operate without the need of any synchronization among nodes. Moreover, the algorithm results in a sparse graph with at most $5n$ edges and a maximum node degree of 10. Existing algorithms with the same promises further require neighbor distance and/or direction information at each node.

## 1 Introduction

We consider $n$ wireless nodes indexed (and uniquely identified) by the natural numbers $1, \ldots, n$ with locations $x_1, \ldots, x_n \in \mathbb{R}^2$. Given $1 \leq i < j \leq n$, Node $i$ may potentially be connected to any Node $j$ with $|x_i - x_j| \leq R$, where $|\cdot|$ is the Euclidean distance, and $R > 0$ is the communication range. As an example, a network consisting of 7 nodes with no connections is as shown in Fig. 1(a). In this example, Node 7 can potentially be connected to any other node except Nodes 2 and 4.

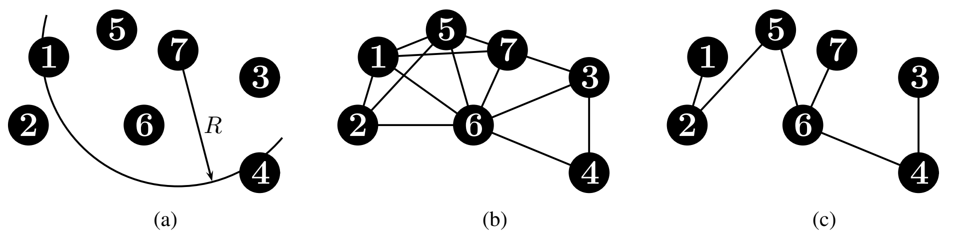


Figure 1: Instances of a network with 7 nodes. Physical locations of the nodes are kept fixed throughout (a)-(c). The "exact" physical location of a given node is the center of the corresponding black disk.

A major goal of topology control is to have a connected network, i.e. a network where there exists a path between any two given nodes so that information from one node may be conveyed to another [1]. In the case of our disk-connectivity model, the network can be made connected only whenever the Gilbert graph $(\mathcal{V}, \mathcal{E}(\mathcal{V}))$ with $\mathcal{V} \triangleq \{1, \ldots, n\}$ and $\mathcal{E}(\mathcal{W}) \triangleq \{(i,j) : i,j \in \mathcal{W},\, i < j,\, |x_i - x_j| \leq R\}$, $\mathcal{W} \subset \mathcal{V}$ is connected. As an example, Fig. 1(b) shows the Gilbert graph corresponding to the setup in Fig. 1(a). This particular graph has 13 edges with a maximum (node) degree of 6, while a general Gilbert graph may have $\frac{1}{2}n(n-1)$ edges with a maximum degree of $n-1$.

The existence of many edges and nodes with high degrees is not desirable in wireless networks due to several practical issues such as radio interference [2, 3]. One thus wishes to obtain sparse connected topologies with a constant maximum node degree. In practice, such a topology should be generated locally with every node picking its own connections according to a certain common algorithm that requires as little information as possible. In this context, given $i \in \mathcal{V}$, let $\mathcal{N}_i \triangleq \{j : j \in \mathcal{V},\, j < i,\, |x_i - x_j| \leq R\}$ represent the *lower neighborhood* of Node $i$ (Lower in the sense that it only contains neighbors with smaller indices/identification numbers.). We assume that a given Node $i$ only knows $\mathcal{N}_i$ and $\mathcal{N}_j$, $j \in \mathcal{N}_i$. In the following, we introduce a corresponding algorithm that provides a connected sparse network with constant maximum degree. We note that several existing local algorithms such as XTC [4], NTC [5], LMST [6], CBTC [7] (also see [8-10] and [11] for a general survey on topology control and other algorithms) can all provide topologies with the same guarantees; however, they further require each node to know its exact distance and/or direction to its neighboring nodes as well as other extra side information. In fact, to the best of our knowledge, no previous algorithm can guarantee even a sparse connected topology (with no degree constraints) under the restrictions that we impose on node knowledge.

# 2 The Algorithm

Given $i \in \mathcal{V}$, consider the Gilbert graph $(\mathcal{N}_i, \mathcal{E}(\mathcal{N}_i))$ generated by the lower neighborhood of Node $i$. It is not difficult to show that $(\mathcal{N}_i, \mathcal{E}(\mathcal{N}_i))$ can have at most 5 connected components, which we shall refer to as $(\mathcal{N}_{ij}, \mathcal{E}(\mathcal{N}_{ij}))$, $j = 1, \ldots, 5$ (Of course, some or all of $\mathcal{N}_{ij}$ may be empty.). Our algorithm (at Node $i$) is then to "Connect to all nodes in the set $\{\max \mathcal{N}_{ij} : \mathcal{N}_{ij} \neq \emptyset\}$." Running the algorithm at each node exactly once results in a graph that we refer to as $(\mathcal{V}, \mathcal{F})$. Nodes may run the algorithm in arbitrary order, or simultaneously in a completely asynchronous fashion.

As an example, for the setup in Fig. 1(a), the algorithm results in the topology in Fig. 1(c). In detail, for Node 6 in Fig. 1(a), the vertex sets of the two connected components induced by $\mathcal{N}_6$ are $\mathcal{N}_{61} = \{1, 2, 5\}$ and $\mathcal{N}_{62} = \{3, 4\}$ so that Node 6 will establish connections to Nodes 5 and 4. Node 1, having no lower neighbors, will not attempt to connect to any other node. On the other hand, for Node 2, we have the single vertex set $\mathcal{N}_{21} = \{1\}$, so that Node 2 will connect to Node 1 (Hence, Node 1 in fact gets connected to Node 2, even though it is not Node 1 that initiates this connection.).

The following theorem summarizes some of the properties of the resulting topology $(\mathcal{V}, \mathcal{F})$

**Theorem 1.** *The graph $(\mathcal{V}, \mathcal{F})$ is connected if and only if the Gilbert graph $(\mathcal{V}, \mathcal{E}(\mathcal{V}))$ is connected. Moreover, we have $|\mathcal{F}| \leq 5n$ and the degree of each node in $(\mathcal{V}, \mathcal{F})$ is no more than $10$.*

*Proof.* For the statement regarding connectivity, we only need to prove the "if" part with the "only if" part being trivial. Suppose $(\mathcal{V}, \mathcal{E}(\mathcal{V}))$ is connected. Then, for any given two nodes in $\mathcal{V}$, there is a (finite) path in $(\mathcal{V}, \mathcal{E}(\mathcal{V}))$ that connects these two nodes with each edge in the path consisting of two neighboring nodes. To show that $(\mathcal{V}, \mathcal{F})$ is connected, it is thus sufficient to show that any two Nodes $i$ and $j$ within range and (without loss of generality) $i < j$ are path-connected in $(\mathcal{V}, \mathcal{F})$. To prove this, first note that if $i = j - 1$, then, by design, Node $j$ initiates a connection to Node $i$ and we are done. Otherwise, $\exists k \in \mathcal{V}$ with $i < k < j$ such that (i) Node $j$ initiates a connection to Node $k$, and (ii) there is a path $\mathcal{P}$ in $(\mathcal{V}, \mathcal{E}(\mathcal{V}))$ connecting Node $k$ to Node $i$ such that the index of each node in $\mathcal{P}$ is no more than $k \leq j - 1$. It is then sufficient to show that any two distinct neighboring nodes that appear in $\mathcal{P}$ are path-connected in $(\mathcal{V}, \mathcal{F})$. On the other hand, to prove this latter claim, it is sufficient to show that any two neighboring Nodes $i'$ and $j'$ with indices $i' < j' \leq j - 1$ are path connected in $(\mathcal{V}, \mathcal{F})$. Hence, any two neighboring Nodes $i$ and $j$ with $i < j$ are path-connected in $(\mathcal{V}, \mathcal{F})$ if either $i = j - 1$ or any two neighboring Nodes $i'$ and $j'$ with $i' < j' \leq j - 1$ are path-connected in $(\mathcal{V}, \mathcal{F})$. This last statement describes a finite descent that immediately leads to the path-connectedness of Nodes $i$ and $j$. This concludes the proof of the claim on connectivity.

We now prove the rest of the claims. The inequality $|\mathcal{F}| \leq 5n$ follows immediately as each node initiates at most 5 connections. We now prove the degree bound. Let $i \in \mathcal{V}$. By design, a node with a lower index ($< i$) cannot initiate a connection to Node $i$. On the other hand, Node $i$ itself initiates at most 5 connections. It is thus sufficient to show that there are at most 5 nodes with a higher index ($> i$) initiating a connection to Node $i$. Assume the contrary and suppose there are 6 or more such nodes. Two of these nodes, say Nodes $j$ and $k$ (with $j < k$ without loss of generality) should then be within communication range as well as being within range of Node $i$. This implies $\{i, j\} \subset \mathcal{N}_{k\ell}$ for some $\ell \in \{1, \ldots, 5\}$ with $i \notin \mathcal{N}_{k\ell'}$ and $j \notin \mathcal{N}_{k\ell'}$ for $\ell' \neq \ell$. Since $\max C_{k\ell} \geq \max\{i, j\} = j > i$, and $i \notin \mathcal{N}_{k\ell'}$ for $\ell' \neq \ell$, we have, in fact, $\max \mathcal{N}_{k\ell} \neq i$ for every $\ell$. This contradicts the fact that Node $k$ initiates a connection to Node $i$ and thus proves the degree bound. $\square$

There is more to say about the algorithm that generates $(\mathcal{V}, \mathcal{F})$. For example, it can be shown that for a uniform random network [12] on $[0, 1]^2$, the algorithm provides asymptotically almost-sure connectivity with $n(1 + o(1))$ edges if $R^2 \in \Omega(\frac{1}{n} \log n)$ is just above the connectivity threshold. There are also applications to interference networks in the spirit of [3]. Also, the stretch factors associated with $(\mathcal{V}, \mathcal{F})$ may be evaluated. Due to lack of space, we shall discuss these issues elsewhere.

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