Variational Inference and Big Data

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Model structure

The class of models we are interested in involves observations, global hidden variables, local hidden variables, and fixed parameters.

1. Global variables: $\beta$
2. Local variables: $z_1, \ldots, z_n$
3. Other fixed parameters: $\alpha$

The local variables govern distributions on the data $x_1, \ldots, x_n$.

Given global variables, what happens locally is independent of each other.
Example: Topic models

- A core idea is of a “topic” — a probability distribution on words.

- Words in a document generated from some combination of topics.

- Text models differ mainly in how they put together topics to build up larger models appropriate for the specific problem goals.
Example topic model: Latent Dirichlet allocation (LDA)

- Draw topics (distributions on words): $\beta_k \sim \text{Dirichlet}, k = 1, \ldots, K$.

- Draw distribution on topics for each document: $\theta_i \sim \text{Dirichlet}$.

- For word $n$ in document $i$,
  - Sample topic indicator $c_{ni} \sim \text{Discrete}(\theta_i)$,
  - Sample word value $x_{ni} \sim \text{Discrete}(\beta_{c_{ni}})$.

The global variables are $\beta = \{\beta_1, \ldots, \beta_K\}$.

The local variables are $z_i = \{\theta_i, c_{1i}, \ldots, c_{ni}\}$. 
Example topics

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<thead>
<tr>
<th>music</th>
<th>book</th>
<th>art</th>
<th>game</th>
<th>show</th>
</tr>
</thead>
<tbody>
<tr>
<td>band</td>
<td>life</td>
<td>museum</td>
<td>knicks</td>
<td>film</td>
</tr>
<tr>
<td>songs</td>
<td>novel</td>
<td>show</td>
<td>nets</td>
<td>television</td>
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<td>story</td>
<td>exhibition</td>
<td>points</td>
<td>movie</td>
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<td>album</td>
<td>books</td>
<td>artist</td>
<td>team</td>
<td>series</td>
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<td>life</td>
</tr>
<tr>
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<td>love</td>
<td>painting</td>
<td>games</td>
<td>man</td>
</tr>
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<td>children</td>
<td>century</td>
<td>night</td>
<td>character</td>
</tr>
<tr>
<td>night</td>
<td>family</td>
<td>works</td>
<td>coach</td>
<td>know</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>theater</th>
<th>clinton</th>
<th>stock</th>
<th>restaurant</th>
<th>budget</th>
</tr>
</thead>
<tbody>
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<td>market</td>
<td>sauce</td>
<td>tax</td>
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<tr>
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<td>campaign</td>
<td>percent</td>
<td>menu</td>
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</tr>
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<td>fund</td>
<td>food</td>
<td>county</td>
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<tr>
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<td>political</td>
<td>investors</td>
<td>dishes</td>
<td>mayor</td>
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<tr>
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<td>republican</td>
<td>funds</td>
<td>street</td>
<td>billion</td>
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<td>dole</td>
<td>companies</td>
<td>dining</td>
<td>taxes</td>
</tr>
<tr>
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<td>presidential</td>
<td>stocks</td>
<td>dinner</td>
<td>plan</td>
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<td>chicken</td>
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<tr>
<td>directed</td>
<td>house</td>
<td>trading</td>
<td>served</td>
<td>fiscal</td>
</tr>
</tbody>
</table>
Model structure

A property these models share is that the joint distribution factorizes over the local variables,

\[ p(x, \beta, z|\alpha) = p(\beta|\alpha) \prod_{i=1}^{n} p(x_i, z_i|\beta, \alpha). \]

Given the global variables, the local variables and corresponding data are conditionally independent.

Our goal is to approximate the posterior distribution of the hidden variables given the observations,

\[ p(\beta, z|x, \alpha) = \frac{p(x, \beta, z|\alpha)}{\int_{\beta, z} p(x, \beta, z|\alpha) d\beta dz}. \]
Variational inference

Mean-field variational Bayes is a deterministic inference method for approximating the posterior distribution of a Bayesian model.

1. Define a factorized distribution to approximate the posterior,

\[ p(\beta, z| x) \approx q(\beta) \prod_i q_i(z_i). \]

2. Lower bound the marginal likelihood,

\[
\ln p(x|\alpha) = \ln \int_{\beta, z} p(x, \beta, z|\alpha) d\beta dz \\
\geq \int_{\beta, z} q(\beta) \prod_i q_i(z_i) \ln \left\{ \frac{p(x, \beta, z|\alpha)}{q(\beta) \prod_i q_i(z_i)} \right\} d\beta dz \\
:= \mathcal{L}.
\]

3. Maximize \( \mathcal{L} \) with respect to parameters of each \( q \). This is equivalent to minimizing the KL divergence between the full posterior and \( q \).
Conjugate models

Conditional independence in the likelihood:

\[ p(w_1, \ldots, w_n | \eta) = \prod_{i=1}^{n} p(w_i | \eta) = \left[ \prod_{j=1}^{n} h(w_j) \right] \exp \left\{ \eta^T \sum_j t(w_j) - nA(\eta) \right\} \]

A conjugate prior for \( \eta \):

\[ p(\eta | \chi, \nu) = f(\chi, \nu) \exp \{ \eta^T \chi - \nu A(\eta) \} \]

Through Bayes rule the posterior is in the same family:

\[ p(\eta | \chi', \nu') = f(\chi', \nu') \exp \{ \eta^T \chi' - \nu' A(\eta) \} \]

with \( \chi' = \chi + \sum_j t(w_j) \) and \( \nu' = \nu + n \).
Define the $q$ distribution of $\eta$ as

$$q(\eta|\chi', \nu') = f(\chi', \nu') \exp\{\eta^T \chi' - \nu' A(\eta)\}.$$ 

Optimize the variational objective wrt parameters $\chi'$ and $\nu'$.

By differentiating variational objective with respect to $[\chi', \nu']^T$, we find

$$\nabla \mathcal{L} = - \begin{bmatrix}
\frac{\partial^2 \ln f(\chi', \nu')}{\partial \chi' \partial \chi'} & \frac{\partial^2 \ln f(\chi', \nu')}{\partial \chi' \partial \nu'} \\
\frac{\partial^2 \ln f(\chi', \nu')}{\partial \nu' \partial \chi'} & \frac{\partial^2 \ln f(\chi', \nu')}{\partial \nu'^2}
\end{bmatrix} \begin{bmatrix}
\chi + \sum_j \mathbb{E}_q t(w_j) - \chi' \\
\nu + n - \nu'
\end{bmatrix}.$$ 

Setting this to zero we can read off the update to $q(\eta|\chi', \nu').$
General form of batch variational inference

For each parameter $\eta_k$ in the conjugate exponential model:

- Define a distribution $q(\eta_k)$ in same family as the prior.
- For each $\eta_k$, update the parameters of $q(\eta_k)$ with the expected sufficient statistics wrt current $q(\eta_j), j \neq k$.

Gibbs sampling: Sample from this distribution using sufficient statistics of the current sample.
Writing out the variational objective

The variational objective for our model framework is

$$\mathcal{L} = \sum_{i=1}^{n} \mathbb{E}_q \ln \{ p(x_i, z_i | \beta, \alpha) / q(z_i) \} + \mathbb{E}_q \ln p(\beta | \alpha) - \mathbb{E}_q \ln q(\beta)$$

Batch algorithm:

1. For each $i$, optimize $q(z_i)$
2. Optimize $q(\beta)$
3. Repeat

When $n$ is huge and a non-trivial amount of work is needed to optimize each $q(z_i)$, Step 1 can take a very long time.
Stochastic variational inference

Because the likelihood factorizes, the objective function splits into a sum over local variable terms,

\[
\mathcal{L} = \sum_{i=1}^{n} \mathbb{E}_q \ln \left\{ p(x_i, z_i | \beta, \alpha) / q(z_i) \right\} + \mathbb{E}_q \ln p(\beta | \alpha) - \mathbb{E}_q \ln q(\beta).
\]

local variables for \( q(z_i) \)

This suggests we can use stochastic optimization to maximize \( \mathcal{L} \).
Stochastic variational inference

Create a new objective function by sampling and scaling,

\[ \mathcal{L}_t = \sum_{i \in C_t} \frac{n}{|C_t|} \mathbb{E}_q \ln \{ p(x_i, z_i | \beta, \alpha) / q(z_i) \} + \mathbb{E}_q \ln p(\beta | \alpha) - \mathbb{E}_q \ln q(\beta). \]

local variables for subset of \( q(z_i) \)

We can show that \( \mathbb{E}[\mathcal{L}_t] = \mathcal{L} \).

Stochastic algorithm:

1. Select a small subset of \( q(z_i) \) at random and optimize
2. Scale up the impact of the subset by \( n / |C_t| \)
3. Take a step in direction of natural gradient for \( q(\beta) \)
4. Repeat
Stochastic variational inference

Take gradient step for the global $q(\beta)$:

$$[\chi', \nu']^T = [\chi'_{\text{old}}, \nu'_{\text{old}}]^T + \rho_t B \nabla \mathcal{L}_t$$

In this case, the gradient is:

$$\nabla \mathcal{L}_t = - \begin{bmatrix}
\frac{\partial^2 \ln f(\chi', \nu')}{\partial \chi' \partial \chi'} & \frac{\partial^2 \ln f(\chi', \nu')}{\partial \chi' \partial \nu'} \\
\frac{\partial^2 \ln f(\chi', \nu')}{\partial \nu' \partial \chi'} & \frac{\partial^2 \ln f(\chi', \nu')}{\partial \nu' \partial \nu'}
\end{bmatrix} \begin{bmatrix}
\chi + \frac{n}{|C_t|} \sum_{j \in C_t} \mathbb{E}_{q_t}(w_j) - \chi' \\
\nu + n - \nu'
\end{bmatrix}.$$
Simplifying inference

Set $B$ to inverse Fisher information of $q(\eta)$ and the left matrix cancels.

\[
\chi' = (1 - \rho_t)\chi'_{\text{old}} + \rho_t \left( \chi + \frac{n}{|C_t|} \sum_{j \in C_t} \mathbb{E}_q t(w_j) \right),
\]

\[
\nu' = (1 - \rho_t)\nu'_{\text{old}} + \rho_t (\nu + n).
\]

If the step size $\rho_t$ is such that \(\sum_t \rho_t = \infty\) and \(\sum_t \rho_t^2 < \infty\), this approach converges to a local optimal of $L$. 
Stochastic variational inference

A Dirichlet-multinomial example:

\[ \theta \sim \text{Dirichlet}(\chi), \quad z_i \overset{iid}{\sim} \text{Multinomial}(\theta), \quad q(\theta) = \text{Dirichlet}(\chi') \]

Where \( z_i \) is a latent indicator variable in a mixture model.

Batch update: \( \chi' = \chi + \sum_i \mathbb{E}_q z_i \)

- Requires update of each \( q(z_i) \).

Stochastic update: \( \chi'_t = (1 - \rho_t)\chi'_{old} + \rho_t (\chi + \frac{n}{|C_t|} \sum_{i \in C_t} \mathbb{E}_q z_i) \)

- Requires update of \( q(z_i) \) only for \( i \in C_t \).
Stochastic vs batch inference using 350K articles from *Nature*. 
Main:


Two applications:
