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Lecture 5: Iterative Combinatorial Auctions

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In this lecture we examine a procedure that generalizes the English auction for a single good to a setting with multiple distinct goods. The model is the same as in the previous lecture, and the goal is also the same: to allocate the items efficiently. Recall that in the English auction, the seller starts with a low price of 0, and increases the price by some small increment at each round. As long as more than one bidder requests the item at the quoted price, the price is increased and the rounds progress. When just a single bidder remains, the auction stops and the bidder is awarded the item at the current price.[1]

Explicitly, the English auction tries to find a *clearing price*: a price at which only one bidder requests the item, so that demand equals supply (because there is just one item). Any price between the highest and second highest values is a clearing price. The English auction converges to the lowest such price (modulo some small increment), which happens to be the winner's VCG payment. This makes it inherit nice incentive properties: it is an optimal strategy to bid for the item when the price is below one's value, and stop bidding when the price exceeds that value.

The properties of the English auction we would like to generalize are

1. Convergence to clearing prices.
2. Convergence to VCG payments.
3. Ascending prices as rounds progress.

We will first formally define "clearing prices" when there is more than one item. Then we will examine a "recipe" to formulate auctions for resource allocation problems. Finally, we will examine the conditions under which iterative combinatorial auctions can reach VCG payments; although the VCG payment is also a clearing price with a single item, we will see that this does not hold in general.

1 Competitive Equilibrium

A sealed-bid auction such as the VCG mechanism of the previous lecture typically charges *payments*: there is one payment from each agent. An iterative auction, on the other hand, quotes *prices*. Prices, in our model, have exactly the same structure as the profile of valuations. Prices are *nonlinear*, meaning they are defined over bundles (the price of a bundle is not necessarily just the sum of the prices of the items it contains). We also consider prices that are *non-anonymous*: different agents may see a different price for the same bundle, a practice sometimes called "price discrimination". Thus for each agent i

[1] In fact, this is more accurately called the "Japanese auction". In the traditional English auction, the bidders themselves quote prices at each round to outbid each other. If we disallow jump bids, so that bidders can only bid a small increment above the current price, the progression is the same as in the Japanese auction.

there is a price function $p_i : 2^M \to \mathbf{R}_+$ defined over bundles. For a bundle S and two distinct agents i, j we may have $p_i(S) \neq p_j(S)$ in general. Like valuations, our prices will be monotone and normalized.

If $p_i = p_j$ for any two agents i and j, the prices are *anonymous*. If the price of each bundle is the sum of the prices of its constituent items, the prices are *linear*:

$$p_i(S) = \sum_{j \in S} p_i(j)$$

(With a slight abuse of notation we write $p_i(j)$ instead of $p_i(\{j\})$.) We identify three orders or pricing [2]:

1. Linear and anonymous.
2. Nonlinear and anonymous.
3. Nonlinear and non-anonymous.

When we speak of "linear prices" from now on, it will be understood that they are also anonymous. Linear prices are usually represented by a vector $p \in \mathbf{R}^m$. (The notation p here is overloaded, since it's sometimes used to described functions over bundles or just a price for each item when prices are linear.) The jth component of p, denoted p_j, is the price of item j, and the price of a bundle S—which does not depend on the agent—is $\sum_{j \in S} p_j$ as just described above.

In the following definition, recall from the previous lecture that N is the set of agents, M is the set of items, and Γ is the set of feasible allocations (those that do not allocate the same item to more than one agent).

Definition 1 *A pair $\langle R, p \rangle$ consisting of an allocation R together with prices $p = (p_i)_{i \in N}$ is a* competitive equilibrium *if the following hold.*

$$v_i(R_i) - p_i(R_i) \geq v_i(R'_i) - p_i(R'_i) \qquad (i \in N, R'_i \subseteq M) \tag{1}$$

$$\sum_{i \in N} p_i(R_i) \geq \sum_{i \in N} p_i(R'_i) \qquad (R' \in \Gamma) \tag{2}$$

In a competitive equilibrium, each agent's allocated bundle maximizes the agent's utility at the given prices, and the chosen allocation also maximizes the seller's revenue at the given prices. In this sense, supply equals demand and the market clears. If $\langle R, p \rangle$ is a competitive equilibrium, we say that prices p *support* R.

Let $\mathcal{E} \subseteq \Gamma$ denote the set of allocations R such that $\langle R, p \rangle$ is a competitive equilibrium for some prices p. Let $\mathcal{P} \subseteq \mathbf{R}^{N \times 2^M}$ denote the set of prices p such that $\langle R, p \rangle$ is a competitive equilibrium for some allocation R. The set of competitive equilibria is in fact a product set.

Proposition 1 *The set of competitive equilibria is $\mathcal{E} \times \mathcal{P}$.*

Proof. Let $\langle R, p \rangle$ be a competitive equilibrium, and let $\langle R', p' \rangle$ be another. We have

$$\begin{aligned}\sum_{i \in N} v_i(R_i) &= \sum_{i \in N} \left[v_i(R_i) - p'_i(R_i)\right] + \sum_{i \in N} p'_i(R_i) \\ &\leq \sum_{i \in N} \left[v_i(R'_i) - p'_i(R'_i)\right] + \sum_{i \in N} p'_i(R'_i) \\ &= \sum_{i \in N} v_i(R'_i).\end{aligned} \tag{3}$$

By an identical argument, $\sum_{i \in N} v_i(R'_i) \leq \sum_{i \in N} v_i(R_i)$, and so $\sum_{i \in N} v_i(R_i) = \sum_{i \in N} v_i(R'_i)$. Hence inequality (3) holds with equality, and each R_i maximizes i's utility at prices p'_i, while R maximizes revenue at prices p'. This shows that $\langle R, p' \rangle$ is a competitive equilibrium. By an identical argument, $\langle R', p \rangle$ is a competitive equilibrium. Therefore the set of competitive equilibria is the product set $\mathcal{E} \times \mathcal{P}$. □

Given this proposition, it makes sense to speak of "competitive equilibrium prices" without reference to any particular allocation; this is in contrast to VCG payments. Like the allocations that result from the VCG mechanism, allocations that arise in competitive equilibrium are efficient.

Theorem 1 *If $\langle R, p \rangle$ is a competitive equilibrium, then R is efficient.*

Proof. Given a feasible allocation R', summing inequalities (1) and (2) yields $\sum_{i \in N} v_i(R_i) \geq \sum_{i \in N} v_i(R'_i)$. Since R' was arbitrary, R is efficient. □

In fact, if an allocation is efficient, then there are prices that support it, as the following theorem shows. Together with Theorem 1, the following implies that $\mathcal{E}$ is exactly the set of efficient allocations.

Theorem 2 *For every efficient allocation R, there exist prices p such that $\langle R, p \rangle$ is a competitive equilibrium.*

Proof. Define prices $p_i(S) = v_i(S)$ for all $S \subseteq M$ and $i \in N$. An agent receives utility 0 from any bundle, so R_i is trivially utility-maximizing to agent i, for all $i \in N$. Since the revenue of an allocation equals its value, R must then maximize revenue because it is efficient. □

The proof of Theorem 2 shows that order 3 competitive equilibrium prices always exist. You can construct counter-examples for yourself to show that lower-order competitive equilibrium prices do not necessarily exist over the domain of general valuations.

2 Designing Iterative Auctions

de Vries *et al.* [3] have observed that many iterative auctions for multiple items—at least those with convergence guarantees—are in fact instances of algorithms known as *primal-dual* or *subgradient* on appropriate linear programs. This powerful observation leads to a fairly straightforward "recipe" for designing an iterative auction, given in Figure 1.
(We will not cover subgradient algorithms; see [1, 4] for such auctions.) In general, this does not give an auction with ascending prices, or that converges to VCG payments. The

1. Formulate the efficient allocation problem as a linear program.
    - The LP should have an integer optimal solution, because the items are indivisible.
    - The dual variables are construed as prices or utilities.
2. Run a primal-dual algorithm on the LP.
3. Interpret the algorithm as an auction.

Figure 1: Recipe for an iterative auction.

primal-dual algorithm has to be tuned to obtain these properties, and this requires some creativity.

In the remainder of these notes, we assume that valuations are integer, $v_i : 2^M \to \mathbf{Z}_+$. Consider the following linear program, the "primal" (P).

$$\begin{aligned} \max_{x \geq 0, z \geq 0} \quad & \sum_{i \in N} \sum_{S \subseteq M} v_i(S) x_i(S) \\ \text{subject to} \quad & x_i(S) = \sum_{R \in \Gamma : R_i = S} z(R) \quad (i \in N, S \subseteq M) && (4) \\ & \sum_{S \subseteq M} x_i(S) = 1 \qquad (i \in N) && (5) \\ & \sum_{R \in \Gamma} z(R) = 1 && (6) \end{aligned}$$

This LP has a variable $0 \leq x_i(S) \leq 1$ to indicate whether bundle S is allocated to agent i, for every $S \subseteq M$ and $i \in N$. It also has a variable $z(R)$ to denote whether feasible allocation R is chosen, for every $R \in \Gamma$. Constraints (5) state that each agent can only be allocated one bundle, and constraints (6) state that only one allocation can be selected. Constraints (4) ensure that the bundles allocated correspond to the allocation selected.

In principle the variables can be fractional, but this LP has an integer optimal solution, where each variable is either 1 or 0. We must allow for fractional solutions rather than just integer solutions, because otherwise fundamental results from linear programming such as strong duality do not hold. On the other hand, we want there to be at least one integer optimal solution, because the integer solutions correspond to feasible allocations of the indivisible items.

The linear program (P) has an exponential number of variables and constraints, so it does not find an efficient allocation in polynomial time. But this is not the point here; we want to demonstrate how a primal-dual algorithm to solve (P) can be construed as an iterative auction.

2.1 The Primal-Dual Auction

The dual to (P) is the following linear program (D).

$$\begin{aligned} \min_{\pi,p} \quad & \sum_{i \in N} \pi_i + \pi_0 & & \\ \text{s.t.} \quad & \pi_i \geq v_i(S) - p_i(S) & (i \in N, S \subseteq M) & \quad (7) \\ & \pi_0 \geq \sum_{i \in N} p_i(R_i) & (R \in \Gamma) & \quad (8) \end{aligned}$$

The variables p here have a natural interpretation as prices. At an optimal solution we clearly have

$$\pi_i = \max_{S \subseteq M} v_i(S) - p_i(S), \tag{9}$$

so π_i can be construed as the maximum utility i can achieve over all bundles given prices p. Similarly, at an optimal solution we have

$$\pi_0 = \max_{R \in \Gamma} \sum_{i \in N} p_i(R), \tag{10}$$

so π_0 can be construed as the maximum revenue achievable over all possible feasible allocations given prices p.

Let $D_i(p) = \arg\max_{S \subseteq M}[v_i(S) - p_i(S)]$ be the set of bundles that maximizes i's utility at prices p, its *demand set*. Let $\Gamma^*(p) = \arg\max_{R \in \Gamma} \sum_{i \in N} p_i(R_i)$ be the set of allocations that maximize the seller's revenue at prices p, its *supply set*. The complementary slackness conditions are

$$x_i(S) > 0 \quad \Rightarrow \quad S \in D_i(p) \tag{11}$$

$$z(R) > 0 \quad \Rightarrow \quad R \in \Gamma^*(p) \tag{12}$$

(There are other complementary slackness conditions, but they automatically hold in this case for feasible primal and dual solutions.) Let (x^*, z^*) and (π^*, p^*) be optimal primal and dual solutions, where the primal solution is integral, thus corresponding to an allocation R. The pair of solutions satisfy the complementary slackness conditions.[2] By (11) we see that R_i maximizes i'as utility at prices p, and by (12) we see that R maximizes the seller's revenue. Thus $\langle R, p \rangle$ is a competitive equilibrium! If we solve the primal and dual programs, obtaining an integer solution for the primal, we obtain an efficient allocation together with clearing prices, exactly as desired.

The primal-dual algorithm tries to find a competitive equilibrium by starting with some candidate prices p, and trying to find an allocation R such that the complementary slackness conditions hold. Taking contrapositives, the conditions are

$$S \notin D_i(p) > 0 \quad \Rightarrow \quad x_i(S) = 0 \tag{13}$$

$$R \notin \Gamma^*(p) > 0 \quad \Rightarrow \quad z(R) = 0 \tag{14}$$

[2] See the LP primer posted on the website for this and other fundamental facts about linear programming. `http://www.cs.columbia.edu/coms6998-3/lpprimer.pdf`

The following "restricted primal" (RP) tries to find a solution that satisfies these conditions, given prices p. Note that only $x_i(S)$ such that $S \in D_i(p)$ are present, because the others are set to 0, and similarly for the z variables.

$$
\begin{aligned}
\max_{x\geq 0, z\geq 0, \delta\geq 0} \quad & -\sum_{i\in N} \delta_i \\
\text{s.t.} \quad x_i(S) = & \sum_{R\in\Gamma^*(p): R_i = S} z(R) \quad (i \in N, S \in D_i(p)) \qquad (15) \\
& \sum_{S\in D_i(p)} x_i(S) + \delta_i = 1 \qquad (i \in N) \\
& \sum_{R\in\Gamma^*(p)} z(R) = 1
\end{aligned}
$$

Like (P), this program has an integer optimal solution, for similar reasons. The variables δ are slack variables.[3] Without them the program might be infeasible, because p are not necessarily clearing prices. If there is a solution to the program such that all slack variables are 0, then we have found primal and dual solutions that satisfy complementary slackness (i.e., a competitive equilibrium), and we are done. Otherwise, the optimal value of the program is negative.

In the latter case, the primal-dual algorithm updates the prices p using the solution to the "dual of the restricted primal" (DRP).

$$
\begin{aligned}
\min_{\lambda,\mu} \quad & \sum_{i\in N} \mu_i + \mu_0 \\
\text{s.t.} \quad & \mu_i \geq -\lambda_i(S) \qquad (i \in N, S \in D_i(p)) \qquad (16) \\
\mu_0 \geq & \sum_{i\in N: R_i \in D_i(p)} \lambda_i(R_i) \quad (R \in \Gamma^*(p)) \qquad (17) \\
& \mu_i \geq -1 \qquad (i \in N) \qquad (18)
\end{aligned}
$$

The variables λ_i here correspond to *changes* in the prices p_i. Similarly μ_i corresponds to a change in the utility of agent i, and μ_0 to a change in the seller's revenue. Let (μ^*, λ^*) be an optimal solution to (DRP). If the (RP) has a negative objective value, then so does (DRP) by strong duality. Assuming $(\pi + \mu^*, p + \lambda^*)$ is feasible for (D), the value of this new solution for (D) is

$$
\begin{aligned}
& \sum_{i\in N} [\pi_i + \mu_i] + [\pi_0 + \mu_0] \\
= & \sum_{i\in N} \pi_i + \pi_0 + \sum_{i\in N} \mu_i + \mu_0 \\
< & \sum_{i\in N} \pi_i + \pi_0
\end{aligned}
$$

[3] Normally the primal-dual algorithm introduces slack variables to all constraints. Here they are added to just some constraints to obtain ascending prices. Note that in this case, the slack variables still ensure feasibility.

Given a set of agents $L \subseteq N$ and demand sets $D_i(p)$ for all $i \in N$, set

$$\lambda_i(S) = \begin{cases} 1 & i \in L, S \in D_i(p) \\ 0 & \text{otherwise} \end{cases}$$
$$\mu_i = \begin{cases} -1 & i \in L \\ 0 & \text{otherwise} \end{cases}$$
$$\mu_0 = |L| - 1$$

Figure 2: Price adjustment.

1. At round $t = 0$, set $p^t = \mathbf{0}$.
2. Collect the demand correspondence $D_i(p^t)$ of each $i \in N$.
3. Solve (RP) to obtain an under-supplied set of agents $L \subseteq N^+$.
   - If N is not under-supplied, output $\langle R^t, p^t \rangle$ where R^t is the allocation corresponding to the solution of (RP).
   - Otherwise, perform the price adjustment on L and go to step 2.

Figure 3: The primal-dual auction.

Thus $(\pi + \mu^*, p + \lambda^*)$ is a strict improvement over (π, p) in (D). Note that we do not really need an optimal solution to (DRP) to get an improvement; any solution with a negative value will do.

In sum, assuming (RP) has negative objective value, we would like to find a feasible solution (μ, λ) to (DRP) such that the value of the solution is negative and $(\pi + \mu, p + \lambda)$ is feasible in (D). To this end we will use the price adjustment procedure in Figure 2.

To finally define the primal-dual auction, we need a couple more definitions. Given a dual solution (π, p), that satisfies (9) and (10), let $N^+ = \{i \in N \mid \pi_i > 0\}$ be the set of *active bidders*, those that can derive positive utility with some bundle. We also the need the following important concept.

Definition 2 *A set of bidders $L \subseteq N$ is* under-supplied *at prices p if there is no $R \in \Gamma^*(p)$ such that $R_i \in D_i(p)$ for each $i \in L$.*

If the optimal value of (RP) is negative, then clearly N itself is under-supplied. Intuitively, if N is under-supplied, then demand exceeds supply and prices should be raised. The primal-dual auction does this, using (RP) to identify a set of bidders whose prices should be raised. In Figure 3, we use the superscript t to refer to the allocation, prices, etc. in round t.

The primal-dual auction is simply an interpretation of the primal-dual algorithm. At step 2, collecting the demand correspondences to construct (RP) corresponds to the agents "bidding" on the bundles they most prefer. If (RP) has a negative value, there is an imbalance of demand and supply. The primal-dual algorithm finds an increment with which to update the dual solution, and this corresponds to incrementing prices in the auction. The algorithm runs until complementary slackness is satisfied, which translates to a competitive equilibrium in the auction.

To prove the correctness of this algorithm, we need to show that: (i) if N is under-supplied, we can find an under-supplied set $L \subseteq N^+$ in step 3, (ii) the price adjustment leads to a new feasible solution for (D), (iii) the prices are ascending in each round, and (iv) the auction terminates in a finite number of steps. These facts are proved in the next section.

2.2 Auction Properties

We write $v \geq p$ for short to mean $v_i(S) \geq p_i(S)$ for all $i \in N$ and $S \subseteq M$.

Lemma 1 *Given prices p such that $v \geq p$, if N is under-supplied (i.e., the value of (RP) is negative), then N^+ is under-supplied.*

Proof. Consider $i \notin N^+$, so that $\pi_i = 0$. This means that $p_i \geq v_i$. But $v_i \geq p_i$ by assumption, so $v_i = p_i$. Hence $D_i(p) = 2^M$; agent i's demand set is the entire set of bundles. Thus whatever allocation is chosen, agent i will receive an element of its demand set. It follows that if N is under-supplied, then $N - i$ is under-supplied, because it is necessarily the agents in $N - i$ whose demands cannot all be simultaneously satisfied by an allocation $R \in \Gamma^*(p)$. Repeating this argument, we conclude that N^+ is under-supplied. □

In light of Lemma 1, we can use $-\sum_{i \in N^+} \delta_i$ in the objective of (RP) to identify an under-supplied $L \subseteq N^+$ to use in step 3 of the primal-dual auction, because we have $\delta_i = 0$ for $i \notin N^+$ at an optimal solution.

Lemma 2 *Given an under-supplied set $L \subseteq N^+$ and integer (π, p), the price adjustment specifies a feasible solution (μ, λ) for (DRP) such that $(\pi + \mu, p + \lambda)$ is feasible for (D).*

Proof. Assume (π, p) are integer. Let (μ, λ) be the solution specified by the price adjustment. Constraints (16) and (18) are clearly satisfied. By the fact that L is under-supplied, the summation

$$\sum_{i \in N : R_i \in D_i(p)} \lambda_i(R_i)$$

is over at most $|L| - 1$ terms. Thus the right-hand side of (17) is at most $|L| - 1$ and this constraint is also satisfied.

Let $(\pi', p') = (\pi + \mu, p + \lambda)$. If $i \notin L$, then $(\pi', p') = (\pi, p)$ so constraint (7) for i still holds. For $i \in L$ there are two cases. If $S \in D_i(p)$, then we have

$$\begin{aligned} \pi_i &= v_i(S) - p_i(S) \\ \pi_i + \mu_i &= v_i(S) - [p_i(S) + \lambda_i(S)] \\ \pi'_i &= v_i(S) - p'_i(S) \end{aligned}$$

where the second equality follows because $-\mu_i = \lambda_i(S) = 1$. Otherwise for $S \notin D_i(p)$ we have

$$\begin{aligned} \pi_i &> v_i(S) - p_i(S) \\ \pi_i + \mu_i &\geq v_i(S) - p_i(S) \\ \pi'_i &= v_i(S) - p'_i(S) \end{aligned}$$

where the second inequality follows from the fact that v, π, and p are integer and $\mu_i = -1$. Finally, note that for any $R \in \Gamma^*(p)$, in the sum $\sum_{i \in N} p'_i(R_i)$ we have $p'_i = p_i + 1$ for at most $|L| - 1$ of the terms by the fact that N is under-supplied, and the remaining are $p'_i = p_i$. As a result $\sum_{i \in N} p'_i(R_i) \leq \sum_{i \in N} p_i(R_i) + |L| - 1 = \pi_0 + \mu_0$. Thus $(\pi + \mu, p + \lambda)$ is feasible for (D). □

To show that prices are ascending, we want to show that $p_i^t(\emptyset) = 0$ in all rounds, or equivalently that $\lambda_i^t(S) = 0$ at every round (because we start with all prices set to 0). The reason for this is that we keep prices normalized, so if $p_i^t(\emptyset)$ were ever to rise above 0, we would subtract $p_i^t(\emptyset)$ from each component of p_i^t to renormalize the prices. Effectively, this means an increase in $p_i^t(\emptyset)$ is in fact a decrease in the price of every other bundle.

Lemma 3 *Given an under-supplied set $L \subseteq N^+$, and assuming $p_i(\emptyset) = 0$ for all $i \in N$, the price adjustment specifies $\lambda_i(\emptyset) = 0$ for all $i \in N$.*

Proof. Assume for the sake of contradiction that $\lambda_i(\emptyset) = 1$ according to the price adjustment. This can only happen if $\emptyset \in D_i(p)$. But in that case $\pi_i = 0$, so $i \notin N^+$. This a contradiction, as we only increase the prices for $i \in L \subseteq N^+$. □

We are now ready to prove correctness of the primal-dual auction.

Theorem 3 *The primal-dual auction converges to a competitive equilibrium in a finite number of steps.*

Proof. If at any round t the set of agents N is not under-supplied, then $\langle R^t, p^t \rangle$ is a competitive equilibrium by complementary slackness, as argued in the previous section.

Assume that in round $t-1$, N is under-supplied. By Lemma 1 we can find an $L \subseteq N^+$ such that L is under-supplied, and the price adjustment is well-defined. Assume further that (π^{t-1}, p^{t-1}) is integer, $v \geq p^{t-1}$, and $p_i^{t-1}(\emptyset) = 0$ for each $i \in N$. Note that these three conditions hold at round 0 where $p^0 = 0$.

By Lemma 2, $(\pi^t, p^t) = (\pi^{t-1} + \mu^{t-1}, p^{t-1} + \lambda^{t-1})$ is a feasible solution to (D). Since (π^{t-1}, p^{t-1}) was integer, so is (π^t, p^t) as the price adjustment is integer. As $\lambda^{t-1}(\emptyset) = 0$ for all $i \in N$ by Lemma 3, $p_i^t(\emptyset) = 0$ for all i as well. As $p_i^t = p_i^{t-1}$ for $i \notin L$, $v_i \geq p_i^t$ for these agents. For $i \in L$, the price is increased only on $S \in D_i(p)$, and for these $v_i(S) > p_i^{t-1}(S)$ because $\pi_i > 0$. Thus $v_i(S) \geq p_i^t(S)$ because v_i, p_i^{t-1}, and the price adjustment are all integer.

Therefore, the price adjustment correctly updates the prices to a new feasible dual solution as long as N is under-supplied. Note that under the price adjustment, at least one component of p^t is increased by 1. Assume for the sake of contradiction that the algorithm never terminates: N is always under-supplied. Then at least one component of the prices must tend to $+\infty$. But this is impossible, because $v \geq p^t$ at each round, and the valuations are finite. □

3 VCG Payments

In the English auction, it happens to be the case that the VCG payment of the winner is also a clearing price. When generalizing, to multiple items this may not be the case. To ensure that it is, we need a condition on agent valuations. Define the *coalitional value function* $w : 2^N \to \mathbf{R}$ as follows:

$$w(L) = \max_{R \in \Gamma} \sum_{i \in L} v_i(R_i),$$

for all $L \subseteq N$. In words, $w(L)$ is the maximum total value that can be attained by distributing the items among agents L.

Definition 3 *The coalitional value function is* submodular *if for $L' \subseteq L \subseteq N$ we have*

$$w(L + j) - w(L) \leq w(L' + j) - w(L')$$

for all $j \in N$.

In words, the marginal value of adding an agent to a set of agents decreases as the underlying set grows larger. Under this condition, there are competitive equilibrium prices such that the price of agent's i allocation, $p_i(R_i)$, is agent i's VCG payment (see [2]). This will in fact follow from Theorem 5 below. More generally, we have the following result, independent of whether w is submodular. (The proof is omitted.)

Theorem 4 *If π_i is the utility of an agent under a competitive equilibrium, and $\hat{\pi}_i$ is the utility of the agent under the VCG mechanism, then $\hat{\pi}_i \geq \pi_i$.*

Note that there is flexibility in the choice of $L \subseteq N^+$ in step 1 of the auction in Figure 3. Since prices are ascending, we intuitively want to make sure that price increases are not too large to ensure that prices do not overshoot VCG payments, given Theorem 4. This motivates the following definition.

Definition 4 *A set of agents $L \subseteq N$ is* minimally under-supplied *if L is under-supplied but $L - i$ is not under-supplied for all $i \in L$.*

Before our final theorem, we need the following interesting property of the primal-dual auction.

Lemma 4 *At each round t, if $p_i^t(S) > 0$, then $S \in D_i(p^t)$.*

Proof. We show that demand sets are weakly increasing as rounds progress. If $i \notin L$, then $D_i(p^{t-1}) = D_i(p^t)$ as p_i does not change. Assume $i \in L$ and let $S \in D_i(p^{t-1})$. If $S' \in D_i(p^{t-1})$, the prices on S and S' each increase by 1 so S is still weakly preferred to S' (in fact they yield the same utility). If $S' \notin D_i(p^{t-1})$, S is initially strictly preferred to S', and thus weakly preferred to S' after the price update. Now consider the first round after which the price of S to i is positive. By the price update rule, S is in i's demand set in this round. As just argued, it remains in the demand set in all future rounds. □

To prove our the main theorem of this section, we need to leverage the concept of a minimally under-supplied set of bidders, as well as the submodularity condition on w.

Theorem 5 *Assume that at each round the primal-dual auction selects a minimally under-supplied $L \subseteq N^+$ for the price adjustment. Then the auction terminates with VCG payments.*

Proof. We will show that $\pi_i^t \geq \hat{\pi}_i$ at each round. Since the auction terminates at a competitive equilibrium, at the final round T we have $\pi_i^T \leq \hat{\pi}_i$ by Theorem 4. It will thus follow that $\pi_i^T = \hat{\pi}_i$, meaning that $v_i(R_i^T) - p^T(R_i^T) = v_i(R_i^T) - \hat{q}_i$, or simply $p^T(R_i^T) = \hat{q}_i$.

Assume for the sake of contradiction that there is some round t such that $\pi_j^{t-1} \geq \hat{\pi}_j$ for all $j \in N$, but $\pi_i^t < \hat{\pi}_i = w(N) - w(N-i)$ for some $i \in N$. Let R be the allocation in round $t-1$, so $R \in \Gamma^*(p^{t-1})$. The allocation satisfies the demand of exactly $|L|-1$ agents in L, since the set L selected is minimally under-supplied. Thus the revenue increase of this allocation at round t is $|L|-1$. As this is the maximum possible revenue increase for any allocation, and R already maximized revenue in round $t-1$, it also does so in round t.

Let $W = \{j \in N \mid p_j^{t-1}(R_j) > 0\}$. Let R' be an efficient allocation among agents $W+i$. By Lemma 4, $R_j \in D_j(p^{t-1})$ and $R_j \in D_j(p^t)$ for $j \in W$. Therefore,

$$\begin{aligned}
\sum_{j \in N} p_j^t(R_j) &= \sum_{j \in W} p_j^t(R_j) \\
&= \sum_{j \in W} [v_j(R_j) - \pi_j^t] \\
&\leq w(W) - \sum_{j \in W} \pi_j^t \\
&< w(W) - \sum_{j \in W} \pi_j^t + [w(N) - w(N-i) - \pi_i^t] \\
&\leq w(W) - \sum_{j \in W+i} \pi_j^t + w(W+i) - w(W) \\
&= w(W+i) - \sum_{j \in W+i} \pi_j^t \\
&= \sum_{j \in W+i} [v_i(R_j') - \pi_j^t] \\
&\leq \sum_{j \in W+i} p_j^t(R_j')
\end{aligned}$$

The first inequality above follows from the definition of w. The second (strict) inequality follows by assumption, and the third inequality follows from the submodularity of w. The entire derivation contradicts the fact that R maximizes revenue at round t. This completes the proof. □

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