1 Preservation of Desiderata

We prove that the restitution model proposed in Reflections on Simultaneous Impact (Section 5) produces feasible post-impact velocities and continues to satisfy the five core desiderata outlined in the same work.

1.1 Feasibility

A feasible post-impact velocity satisfies \( G(q)^T q^+ \geq 0 \).

**Theorem.** Interpolation yields a feasible post-impact velocity for all coefficients of restitution \( c_r \in [0, 1] \).

**Proof.** Computing the post-impact relative velocity, we obtain:
\[
G^T q^+ = (1 - c_r) G^T q_0^+ + c_r G^T q_1^+.
\]
By construction the LCP model guarantees that \( G^T q_0^+ \geq 0 \). Similarly, upon termination the GR model guarantees that \( G^T q_1^+ \geq 0 \). Each term in this sum is non-negative. Therefore the interpolation yields a feasible velocity.

1.2 Conservation of Momentum

We begin with the observation that interpolating two post-impact velocities is equivalent to interpolating the corresponding impulses.

**Lemma.** Interpolating \( q_0^+ \) and \( q_1^+ \) is equivalent to interpolating \( \lambda_0 \) and \( \lambda_1 \).

**Proof.**
\[
\begin{align*}
\dot{q}^+ &= (1 - c_r) q_0^+ + c_r q_1^+ \\
&= (1 - c_r) (\dot{q}^- + M^{-1} G \lambda_0) + c_r (\dot{q}^- + M^{-1} G \lambda_1) \\
&= \dot{q}^- + M^{-1} G ((1 - c_r) \lambda_0 + c_r \lambda_1)
\end{align*}
\]
Therefore the net impulse magnitude is \( \lambda = (1 - c_r) \lambda_0 + c_r \lambda_1 \).

**Theorem.** Interpolation conserves momentum.

**Proof.** The generalized normals, by construction, conserve momentum and angular momentum, therefore \( G \lambda \) exerts a momentum conserving impulse on the system for any given set of magnitudes \( \lambda \). The interpolated response thus conserves momentum.

1.3 One-Sided

A one-sided impulse satisfies \( \lambda \geq 0 \).

**Theorem.** Interpolation produces one-sided impulses for all \( c_r \in [0, 1] \).

**Proof.** Given two sets of one-sided impulses \( \lambda_0 \geq 0 \) and \( \lambda_1 \geq 0 \), the sum \( (1 - c_r) \lambda_0 + c_r \lambda_1 \geq 0 \) is also one-sided.

1.4 Bounded Kinetic Energy

The post-impact kinetic energy is given by
\[
T(c_r) = \frac{1}{2} ((1 - c_r) \dot{q}_0^+ + c_r \dot{q}_1^+)^T M ((1 - c_r) \dot{q}_0^+ + c_r \dot{q}_1^+).
\]

**Theorem.** Interpolating post-impact velocities from an inelastic and from an elastic response yields a post-impact kinetic energy bounded by that of elastic response.

**Proof.** The kinetic energy is quadratic in \( c_r \) and \( T(0) < T(1) \). Therefore, if the second derivative of the energy with respect to \( c_r \) is positive, the energy can never exceed that of the elastic response when \( c_r \in [0, 1] \). Computing the second derivative, we find that
\[
\frac{\partial^2 T}{\partial c_r^2} = (\dot{q}_1^+ - \dot{q}_0^+)^T M (\dot{q}_1^+ - \dot{q}_0^+).
\]
\( M \) is positive definite, which implies that the second derivative is positive. Therefore, the post-impact kinetic energy is bounded by that of the elastic response.

1.5 Preservation of Symmetry

The interpolation model does not act on the configuration \( q \) of the system, therefore we only consider its effect on the system’s velocity \( \dot{q} \).

**Theorem.** Interpolation preserves symmetry.

**Proof.** Let \( S(q) = q \) define a (potentially non-linear) symmetry in the system’s configuration. This map operates linearly on the velocity as \( \nabla S(q) \dot{q} = \dot{q} \). Given two velocities that respect this symmetry, we find for the interpolant:
\[
\nabla S \dot{q}^+ = \nabla S ((1 - c_r) \dot{q}_0^+ + c_r \dot{q}_1^+) \\
= (1 - c_r) \nabla S \dot{q}_0^+ + c_r \nabla S \dot{q}_1^+ \\
= (1 - c_r) \dot{q}_0^+ + c_r \dot{q}_1^+ \\
= \dot{q}^+
\]
Therefore, the interpolated response preserves symmetry.

1.6 Break-Away

**Theorem.** If a post-impact velocity satisfies \( \nabla g(q)^T \dot{q}^+ > 0 \) under GR, then the interpolated post-impact velocity satisfies \( \nabla g(q)^T \dot{q}^+ > 0 \).

**Proof.** Under interpolation with the inelastic LCP response \( \nabla g^T \dot{q}_0^+ \geq 0 \), we find that
\[
\nabla g^T \dot{q}^+ = (1 - c_r) \nabla g^T \dot{q}_0^+ + c_r \nabla g^T \dot{q}_1^+ > 0
\]
for all \( c_r \in (0, 1] \).