Conservation of Core Desiderata under Interpolation

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1 Preservation of Desiderata

We prove that the restitution model proposed in Reflections on Simultaneous Impact (Section 5) produces feasible post-impact velocities and continues to satisfy the five core desiderata outlined in the same work.

1.1 Feasibility

A feasible post-impact velocity satisfies $\mathsf{G}(\mathsf{q})^T \dot{\mathsf{q}}^+ \geq 0$.

> **Theorem.** Interpolation yields a feasible post-impact velocity for all coefficients of restitution $\mathsf{c}_r \in [0, 1]$.

Proof. Computing the post-impact relative velocity, we obtain:

$$\mathsf{G}^T \dot{\mathsf{q}}^+ = (1 - \mathsf{c}_r)\, \mathsf{G}^T \dot{\mathsf{q}}_0^+ + \mathsf{c}_r \mathsf{G}^T \dot{\mathsf{q}}_1^+$$

By construction the LCP model guarantees that $\mathsf{G}^T \dot{\mathsf{q}}_0^+ \geq 0$. Similarly, upon termination the GR model guarantees that $\mathsf{G}^T \dot{\mathsf{q}}_1^+ \geq 0$. Each term in this sum is non-negative. Therefore the interpolation yields a feasible velocity. □

1.2 Conservation of Momentum

We begin with the observation that interpolating two post-impact velocities is equivalent to interpolating the corresponding impulses.

> **Lemma.** Interpolating $\dot{\mathsf{q}}_0^+$ and $\dot{\mathsf{q}}_1^+$ is equivalent to interpolating λ_0 and λ_1.

Proof.

$$\begin{aligned}
\dot{\mathsf{q}}^+ &= (1 - \mathsf{c}_r)\, \dot{\mathsf{q}}_0^+ + \mathsf{c}_r \dot{\mathsf{q}}_1^+ \\
&= (1 - \mathsf{c}_r) \left(\dot{\mathsf{q}}^- + \mathsf{M}^{-1}\mathsf{G}\lambda_0\right) + \mathsf{c}_r \left(\dot{\mathsf{q}}^- + \mathsf{M}^{-1}\mathsf{G}\lambda_1\right) \\
&= \dot{\mathsf{q}}^- + \mathsf{M}^{-1}\mathsf{G}\left((1 - \mathsf{c}_r)\, \lambda_0 + \mathsf{c}_r \lambda_1\right)
\end{aligned}$$

Therefore the net impulse magnitude is $\lambda = (1 - \mathsf{c}_r)\, \lambda_0 + \mathsf{c}_r \lambda_1$. □

> **Theorem.** Interpolation conserves momentum.

Proof. The generalized normals, by construction, conserve momentum and angular momentum, therefore $\mathsf{G}\lambda$ exerts a momentum conserving impulse on the system for any given set of magnitudes λ. The interpolated response thus conserves momentum. □

1.3 One-Sided

A one-sided impulse satisfies $\lambda \geq 0$.

> **Theorem.** Interpolation produces one-sided impulses for all $\mathsf{c}_r \in [0, 1]$.

Proof. Given two sets of one-sided impulses $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$, the sum $(1 - \mathsf{c}_r)\, \lambda_0 + \mathsf{c}_r \lambda_1 \geq 0$ is also one-sided. □

1.4 Bounded Kinetic Energy

The post-impact kinetic energy is given by

$$T(\mathsf{c}_r) = \frac{1}{2} \left((1 - \mathsf{c}_r)\, \dot{\mathsf{q}}_0^+ + \mathsf{c}_r \dot{\mathsf{q}}_1^+\right)^T \mathsf{M} \left((1 - \mathsf{c}_r)\, \dot{\mathsf{q}}_0^+ + \mathsf{c}_r \dot{\mathsf{q}}_1^+\right).$$

> **Theorem.** Interpolating post-impact velocities from an inelastic and from an elastic response yields a post-impact kinetic energy bounded by that of elastic response.

Proof. The kinetic energy is quadratic in c_r and $T(0) < T(1)$. Therefore, if the second derivative of the energy with respect to c_r is positive, the energy can never exceed that of the elastic response when $\mathsf{c}_r \in [0, 1]$. Computing the second derivative, we find that

$$\frac{\partial^2 T}{\partial {\mathsf{c}_r}^2} = \left(\dot{\mathsf{q}}_1^+ - \dot{\mathsf{q}}_0^+\right)^T \mathsf{M} \left(\dot{\mathsf{q}}_1^+ - \dot{\mathsf{q}}_0^+\right).$$

M is positive definite, which implies that the second derivative is positive. Therefore, the post-impact kinetic energy is bounded by that of the elastic response. □

1.5 Preservation of Symmetry

The interpolation model does not act on the configuration q of the system, therefore we only consider its effect on the system's velocity $\dot{\mathsf{q}}$.

> **Theorem.** Interpolation preserves symmetry.

Proof. Let $S(\mathsf{q}) = \mathsf{q}$ define a (potentially non-linear) symmetry in the system's configuration. This map operates linearly on the velocity as $\nabla \mathsf{S}(\mathsf{q})\dot{\mathsf{q}} = \dot{\mathsf{q}}$. Given two velocities that respect this symmetry, we find for the interpolant:

$$\begin{aligned}
\nabla \mathsf{S} \dot{\mathsf{q}}^+ &= \nabla \mathsf{S} \left((1 - \mathsf{c}_r)\, \dot{\mathsf{q}}_0^+ + \mathsf{c}_r \dot{\mathsf{q}}_1^+\right) \\
&= (1 - \mathsf{c}_r)\, \nabla \mathsf{S} \dot{\mathsf{q}}_0^+ + \mathsf{c}_r \nabla \mathsf{S} \dot{\mathsf{q}}_1^+ \\
&= (1 - \mathsf{c}_r)\, \dot{\mathsf{q}}_0^+ + \mathsf{c}_r \dot{\mathsf{q}}_1^+ \\
&= \dot{\mathsf{q}}^+
\end{aligned}$$

Therefore, the interpolated response preserves symmetry. □

1.6 Break-Away

> **Theorem.** If a post-impact velocity satisfies $\nabla g(\mathsf{q})^T \dot{\mathsf{q}}_1^+ > 0$ under GR, then the interpolated post-impact velocity satisfies $\nabla g(\mathsf{q})^T \dot{\mathsf{q}}^+ > 0$.

Proof. Under interpolation with the inelastic LCP response $\nabla g^T \dot{\mathsf{q}}_0^+ \geq 0$, we find that

$$\nabla g^T \dot{\mathsf{q}}^+ = (1 - \mathsf{c}_r)\, \nabla g^T \dot{\mathsf{q}}_0^+ + \mathsf{c}_r \nabla g^T \dot{\mathsf{q}}_1^+ > 0$$

for all $\mathsf{c}_r \in (0, 1]$. □