Discrete Viscous Sheets: Supplemental Material

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1 Derivation of Analytical Solutions

1.1 Inflation of a spherical viscous sheet.

The inflation of a spherical viscous balloon under internal pressure was used to validate our stretching forces; the analytical formula for the expansion velocity $\dot{r}$ of the balloon is justified here. Because of the symmetry, the strain rate, measured in the plane tangent to the surface, is isotropic: $\dot{\epsilon}_{\alpha\beta} = \frac{\dot{r}}{r}\,\delta_{\alpha\beta}$. Using our constitutive law for stretching, $N_{\alpha\beta} = 2\,\mu\,h\,(\dot{\epsilon}_{\alpha\beta} + \delta_{\alpha\beta}\,\mathrm{Tr}\,\dot{\boldsymbol{\epsilon}})$, this strain gives rise to a membrane stress $N_{\alpha\beta} = 6\,\mu\,h\,\frac{\dot{r}}{r}\,\delta_{\alpha\beta}$. The validation test is set up in conditions such that inertia is negligible. Then the expansion velocity is found by balancing the power dissipated by viscous stress, $\iint N_{\alpha\beta}\dot{\epsilon}_{\alpha\beta}\,da = 4\pi\,r^2\ \times 2 \times 6\,\mu\,h\,\left(\frac{\dot{r}}{r}\right)^2$, with the power of the pressure force, $4\pi\,r^2\,p\,\dot{r}$. This yields $\dot{r} = \frac{p\,r^2}{12\,\mu\,h}$.

1.2 Contraction of a spherical sheet under surface tension.

The contraction of an inviscid spherical viscous sheet was used to validate surface tension force. We derive an analytical expression for the time evolution of such a sheet under surface tension as follows. The normal force density on a spherical surface due to surface tension is $F = \frac{2\gamma}{r}$, which we double to account for the inner and outer surfaces. The rate of change of the sheet's momentum is $\rho h \ddot{r}$, so Newton's second law gives $\frac{4\gamma}{r} = \rho h \ddot{r}$. Conservation of volume in the spherical sheet dictates $h = h_0 \frac{r_0^2}{r^2}$, and substituting this expression for h yields a second-order linear ODE: $T^2\,\frac{\ddot{r}}{r_0} + \frac{r}{r_0} = 0$ for $T = \frac{r_0}{2}\sqrt{\rho h_0/\gamma}$. For initial conditions $r(0) = r_0,\ \dot{r}(0) = 0$, we find that $r(t) = r_0 \cos{(t/T)}$.

2 Derivation of Surface Tension Forces and Jacobians

2.1 Surface Energy

The surface energy E of the liquid sheet is equal to twice the integral of the surface area scaled by the surface tension coefficient γ.

$$E = \iint_{\partial\Omega} 2\gamma\,dA \approx \sum_i 2\gamma A_i \tag{1}$$

where A_i indicates the area of triangle i, and $\partial\Omega$ indicates the boundary of the liquid domain (i.e. the surface). As usual, we compute the force from the potential energy as $F = -\nabla E$, and the force Jacobian as $\frac{\partial F}{\partial X} = -\nabla^2 E$. Since we assume γ is constant, we will ultimately need only the derivatives of triangle area, which we outline below.

The area A of a triangle with vertices $\vec{a}, \vec{b}, \vec{c}$ is typically computed as:

$$A = \frac{1}{2}\|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\| \tag{2}$$

or in indicial notation:

$$A = \frac{1}{2}\sqrt{\epsilon_{ijk}(b_j - a_j)(c_k - a_k)\epsilon_{imn}(b_m - a_m)(c_n - a_n)} \tag{3}$$

We proceed to simplify this expression:

$$A = \frac{1}{2}\sqrt{\epsilon_{ijk}\epsilon_{imn}(b_j - a_j)(c_k - a_k)(b_m - a_m)(c_n - a_n)} \tag{4}$$

$$= \frac{1}{2}\sqrt{(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})(b_j - a_j)(c_k - a_k)(b_m - a_m)(c_n - a_n)} \tag{5}$$

$$= \frac{1}{2}\sqrt{(b_j - a_j)(c_k - a_k)(b_j - a_j)(c_k - a_k) - (b_j - a_j)(c_k - a_k)(b_k - a_k)(c_j - a_j)} \tag{6}$$

For convenience we define:

$$K = (b_j - a_j)(c_k - a_k)(b_j - a_j)(c_k - a_k) - (b_j - a_j)(c_k - a_k)(b_k - a_k)(c_j - a_j) \tag{7}$$

so we have just

$$A = \frac{1}{2}\sqrt{K} \tag{8}$$

2.2 Area Gradient

Next we want to compute gradients of area with respect to triangle vertex positions. We'll first consider a generic vertex $\vec{X}$. Then:

$$\frac{\partial A}{\partial X_i} = \frac{1}{4\sqrt{K}} \frac{\partial K}{\partial X_i} \tag{9}$$

Now we compute $\frac{\partial}{\partial X_i} K$ for each vertex $\vec{X}$.

2.2.1 Derivatives w.r.t. triangle vertices

$$\frac{\partial}{\partial c_i} K = \frac{\partial}{\partial c_i} ((b_j - a_j)(c_k - a_k)(b_j - a_j)(c_k - a_k) - (b_j - a_j)(c_k - a_k)(b_k - a_k)(c_j - a_j))$$

$$\begin{aligned}\frac{\partial}{\partial c_i} K = {} & 2(b_j - a_j)(c_k - a_k)(b_j - a_j) \frac{\partial}{\partial c_i}(c_k - a_k) \\ & -(b_j - a_j)(c_k - a_k)(b_k - a_k) \frac{\partial}{\partial c_i}(c_j - a_j) \\ & -(b_j - a_j)(b_k - a_k)(c_j - a_j) \frac{\partial}{\partial c_i}(c_k - a_k)\end{aligned}$$

Converting derivatives to deltas:

$$\frac{\partial}{\partial c_i} K = 2(b_j - a_j)(c_k - a_k)(b_j - a_j)\delta_{ik} - (b_j - a_j)(c_k - a_k)(b_k - a_k)\delta_{ij} - (b_j - a_j)(b_k - a_k)(c_j - a_j)\delta_{ik}$$

Eliminating deltas:

$$\frac{\partial}{\partial c_i} K = 2(b_j - a_j)(c_i - a_i)(b_j - a_j) - (b_i - a_i)(c_k - a_k)(b_k - a_k) - (b_j - a_j)(b_i - a_i)(c_j - a_j)$$

Relabelling dummy indices, and combining the last two terms we finally get:

$$\begin{aligned}\frac{\partial}{\partial c_i} K &= 2(c_i - a_i)(b_j - a_j)(b_j - a_j) - 2(b_i - a_i)(b_j - a_j)(c_j - a_j) \\ \frac{\partial}{\partial c} K &= 2(\|\vec{b} - \vec{a}\|^2 (\vec{c} - \vec{a}) - ((\vec{c} - \vec{a}) \cdot (\vec{b} - \vec{a}))(\vec{b} - \vec{a}))\end{aligned}$$

By symmetry b will have a similar form...

$$\begin{aligned}\frac{\partial}{\partial b_i} K &= 2(b_i - a_i)(c_j - a_j)(c_j - a_j) - 2(c_i - a_i)(b_j - a_j)(c_j - a_j) \\ \frac{\partial}{\partial b} K &= 2(\|\vec{c} - \vec{a}\|^2 (\vec{b} - \vec{a}) - ((\vec{b} - \vec{a}) \cdot (\vec{c} - \vec{a}))(\vec{c} - \vec{a}))\end{aligned}$$

This just leaves a. Looking carefully at the derivatives we can see another symmetry: previously, where each derivative by c_i of an expression having the form $(c_j - a_j)$ gave a δ_{ij}, we now get a $-\delta_{ij}$. Likewise where each derivative by b_i of $(b_j - a_j)$ gave a δ_{ij}, we now get a $-\delta_{ij}$. Thus we find that:

$$\frac{\partial}{\partial a_i} K = -\left(\frac{\partial}{\partial b_i} K + \frac{\partial}{\partial c_i} K \right) \tag{10}$$

(Note that it is also possible to exploit purely geometric arguments and arrive at equivalent expressions.)

2.3 Area Hessian

Now we need to take the derivatives of the above gradients to compute the Hessian. The general form will be:

$$\frac{\partial^2 A}{\partial X_i \partial Y_k} = -\frac{1}{8K^{\frac{3}{2}}} \frac{\partial K}{\partial X_i} \frac{\partial K}{\partial Y_k} + \frac{1}{4K^{\frac{1}{2}}} \frac{\partial^2 K}{\partial X_i \partial Y_k} \tag{11}$$

The only additional work we need to do is determine $\frac{\partial^2 K}{\partial X_i \partial Y_k}$ for all pairs of vertices X_i and Y_k. We will again be able to exploit symmetry to simplify some terms.

2.3.1 Derivatives of $\frac{\partial K}{\partial b_i}$

Let's start with b_i.

$$
\begin{aligned}
\frac{\partial^2 K}{\partial b_i \partial c_k} &= 2\left(2(b_i - a_i)(c_j - a_j)\delta_{jk} - (c_i - a_i)(b_j - a_j)\delta_{jk} - \delta_{ik}(b_j - a_j)(c_j - a_j)\right) \\
&= 2\left(2(b_i - a_i)(c_k - a_k) - (c_i - a_i)(b_k - a_k) - (b_j - a_j)(c_j - a_j)\delta_{ik}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 K}{\partial b_i \partial b_k} &= 2\left(\delta_{ik}(c_j - a_j)(c_j - a_j) - (c_i - a_i)(c_j - a_j)\delta_{jk}\right) \\
&= 2\left((c_j - a_j)(c_j - a_j)\delta ik - (c_i - a_i)(c_k - a_k)\right)
\end{aligned}
$$

Using symmetry again, we have:

$$
\frac{\partial^2 K}{\partial b_i \partial a_k} = -\left(\frac{\partial^2 K}{\partial b_i \partial b_k} + \frac{\partial^2 K}{\partial b_i \partial c_k}\right) \tag{12}
$$

2.4 Derivatives of $\frac{\partial K}{\partial c_i}$

First we have:

$$
\begin{aligned}
\frac{\partial^2 K}{\partial c_i \partial b_k} &= 2\left(2(c_i - a_i)(b_j - a_j)\delta_{jk} - (b_i - a_i)(c_j - a_j)\delta_{jk} - \delta_{ik}(c_j - a_j)(b_j - a_j)\right) \\
&= 2\left(2(c_i - a_i)(b_k - a_k) - (b_i - a_i)(c_k - a_k) - (c_j - a_j)(b_j - a_j)\delta_{ik}\right)
\end{aligned}
$$

Next:

$$
\begin{aligned}
\frac{\partial^2 K}{\partial c_i \partial c_k} &= 2\left(\delta_{ik}(b_j - a_j)(b_j - a_j) - (b_i - a_i)(b_j - a_j)\delta jk\right) \\
&= 2\left((b_j - a_j)(b_j - a_j)\delta_{ik} - (b_i - a_i)(b_k - a_k)\right)
\end{aligned}
$$

And finally:

$$
\frac{\partial^2 K}{\partial c_i \partial a_k} = -\left(\frac{\partial^2 K}{\partial c_i \partial b_k} + \frac{\partial^2 K}{\partial c_i \partial c_k}\right) \tag{13}
$$

2.5 Derivatives of $\frac{\partial K}{\partial a_i}$

$$
\frac{\partial^2 K}{\partial a_i \partial a_k} = -\left(\frac{\partial^2 K}{\partial b_i \partial a_k} + \frac{\partial^2 K}{\partial c_i \partial a_k}\right) \tag{14}
$$

$$
\frac{\partial^2 K}{\partial a_i \partial b_k} = -\left(\frac{\partial^2 K}{\partial b_i \partial b_k} + \frac{\partial^2 K}{\partial c_i \partial b_k}\right) \tag{15}
$$

$$
\frac{\partial^2 K}{\partial a_i \partial c_k} = -\left(\frac{\partial^2 K}{\partial b_i \partial c_k} + \frac{\partial^2 K}{\partial c_i \partial c_k}\right) \tag{16}
$$