RECURRENCES FOR THE GENUS POLYNOMIALS OF LINEAR SEQUENCES OF GRAPHS

YICHAO CHEN, JONATHAN L. GROSS, TOUFIK MANSOUR, AND THOMAS W. TUCKER

Abstract. In this paper, we develop and exploit the theorem that the sequence of genus polynomials of any *H-linear family* $\{G_n\}$ of graphs satisfies a k^{th}-order homogeneous linear recurrence system $\mathcal{R}$ for some $k \geq 1$, which is proved using the Cayley-Hamilton theorem. One way to derive the coefficients for the recurrence $\mathcal{R}$ is based on the identification of *imbedding types* and on the characteristic polynomial of the associated *production matrix*. In another method, which does not involve a production matrix, the coefficients of $\mathcal{R}$ are calculated as the solution set to a system of k linear equations that can be formed from the initial values of the genus polynomials for $\{G_n\}$. A computational benefit of using the recurrence $\mathcal{R}$ to calculate the next genus polynomial in the sequence, instead of using a production matrix, is that instead of needing k^2 multiplications of polynomials, we now need only k such multiplications. Moreover, having quick access to the linear recursion $\mathcal{R}$ can facilitate proofs of real-rootedness and log-concavity of the polynomials. We illustrate with examples.

1. Introduction

Let G be a graph, possibly with multi-edges and loops. For $i = 0, 1, 2, \dots$, let $g_i(G)$ denote the number of combinatorially distinct cellular imbeddings of G in the orientable surface S_i. Then the generating function

$$g_0(G) + g_1(G)z + g_2(G)z^2 + \cdots + g_{\gamma_{\max}(G)}(G)z^{\gamma_{\max}(G)}$$

is called the ***genus polynomial*** of G and is denoted by $\Gamma_G(z)$. It is assumed that the reader is familiar with the basics of topological graph theory, as found in Gross and Tucker [GT87]. The degree of the genus polynomial equals the maximum genus $\gamma_{\max}(G)$ of the graph G. We observe that the genus polynomial of a graph is an invariant of its homeomorphism type.

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By a *linear sequence of graphs*, we mean, roughly, a sequence $\{G_n\}$ with a specified initial graph G_0 and where G_i is obtained from G_{i-1} by pasting a copy of some fixed graph H to G_{i-1}. When the graph H has been specified, we may call the sequence an *H-linear sequence.* A more precise definition appears below.

A finite sequence $a_0, a_1, \ldots, a_n$ of non-negative real numbers is said to be ***log-concave*** if for every index i such that $0 < i < n$, we have

$$a_{i-1}a_{i+1} \leq a_i^2.$$

A polynomial is said to be log-concave if its sequence of coefficients is log-concave. The ***log-concavity conjecture for genus polynomials*** [GRT89] asserts that every graph genus polynomial is log-concave.

Log-concavity and real-rootedness of polynomials are well-established topics of interest among algebraic combinatorialists. See, for example, [St89, Bre89, GMTW15, Las02, HuKa12].

Many of the papers on genus polynomials are concerned with the derivation of recursions for the genus polynomials of the graphs in linear sequences and proofs of log-concavity. Usually such derivations have been based on *partial genus polynomials*, as defined in Section 2, and a system of simultaneous recursions for them.

This paper demonstrates how, via application of the Cayley-Hamilton theorem, we can derive a linear recursion with coefficients in the polynomial ring $\mathbb{Z}[z]$, for the sequence of full genus polynomials, without explicit derivation of the solutions to the simultaneous recursions for the partial genus polynomials.

This yields a new paradigm for proving log-concavity, based on the linear recursion for the sequence $\{\Gamma_{G_n}(z)\}$ of full genus polynomials of $\{G_n\}$. We observe that the proofs of log-concavity here are shorter and less complicated than typical proofs of log-concavity based on partial genus polynomials (e.g., [GMT14]).

1.1. **Formal definition of an H-linear sequence.** Let H be a graph whose root-vertices are bi-partitioned into a ***rear subset*** $U = \{u_1, u_2, \ldots, u_k\}$ and a ***front subset*** $V = \{v_1, v_2, \ldots, v_k\}$, both of the same cardinality k. For $i = 1, 2, \ldots$, let H_i be an isomorphic copy of H, and let $f_i : H \to H_i$ be an isomorphism. For each $i \geq 1$ and j such that $1 \leq j \leq k$, let $u_{i,j} = f_i(u_j)$ and let $v_{i,j} = f_i(v_j)$. We define an (H, U, V)***-linear sequence of graphs***, or (H, u, v)***-linear***, when U and V are singleton sets $\{u\}$ and $\{v\}$, respectively by recursive amalgamation of some root-vertices:

- Let G_0 be a graph with root-vertices $v_{0,1}, v_{0,2}, \ldots, v_{0,k}$.
- For $i \geq 1$, the graph G_i is constructed from the graph G_{i-1} by amalgamating the vertex $v_{i-1,j}$ of G_{i-1} with the vertex $u_{i,j}$ of (H_i, U_i, V_i), for $j = 1, 2, \ldots, k$.

Either when the designation of root-vertices is implicitly understood, or when it does not matter, we may refer to an ***H-linear sequence of graphs***.

Remark. The term "H-linear" was first used by [Stah91], based on examples in [FGS89]. The construction of H-linear families given here, where front vertices are amalgamated to back vertices, is not the same as in [GKMT18], where edges are added between front roots and back roots. However, it is not hard to interpret either construction in terms of the other.

Example 1.1. Figure 1.1 illustrates a K_4-linear sequence of graphs. The upper row contains both the initial graph G_0, which we observe is homeomorphic to K_4 (so it has the same genus polynomial), and the iterated graph H, which is also homeomorphic to K_4. The lower row illustrates the graph G_3.

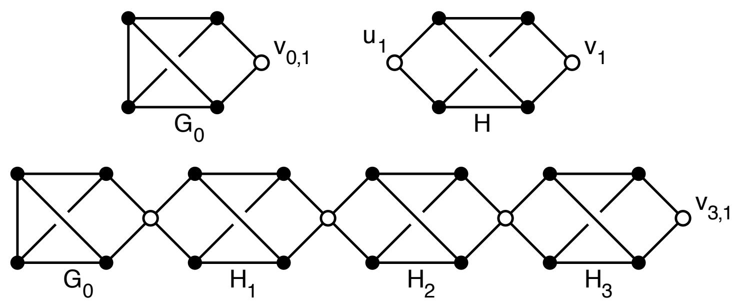

Figure 1.1. A K_4-linear sequence of graphs.

The recurrence for the sequence of genus polynomials of the K_4-linear sequence illustrated in Figure 1.1 is

$$\begin{aligned}
\Gamma_{G_n}(z) &= (84z+8)\Gamma_{G_{n-1}}(z) + (768z^3 - 384z^2)\Gamma_{G_{n-2}}(z) \\
\Gamma_{G_0}(z) &= 14z+2 \\
\Gamma_{G_1}(z) &= 128z^3 + 1112z^2 + 280z + 16.
\end{aligned} \tag{1.1}$$

The two initial conditions are given in Table 4.2 of [GKP10]. The recursion formula is derived here in Section 3, using Theorem 3.3.

To verify the accuracy of our calculations throughout, we have used MAPLE. For relatively small graphs, it is possible, albeit tedious, to calculate genus polynomials and production matrices by pencil and paper. We have used two computer programs written by Imran Khan, one for the calculation of genus polynomials of graphs (based on the "brute-force" Heffter-Edmonds algorithm) and the other for calculating a production matrix, based on the representation of imbedding types as strings of root-vertices [Gro14, GKMT18].

In Section 2, we review imbedding types and production matrices for a linear family of graphs. In Section 3, we show how the Cayley-Hamilton theorem leads us from a production matrix to a linear recurrence for the genus polynomials of the given linear family. Additionally, we give a different method for calculating that linear recurrence, in which there is no need to calculate the production matrix. In Sections 4 and 5, we derive upper bounds for the degrees of the coefficients of the preceding genus polynomials in the main recurrence, for the special case of two 2-valent root-vertices, and we derive the recurrences for two infinite varieties of linear families, and we prove that the genus polynomials for all of their graphs are log-concave. In Section 6, we generalize and strengthen the upper bounds from Section 4, when the iterated graph H is upper-imbeddable; we also derive some bounds for maximum genus that concern the graphs in H-linear sequences. In Section 7, we present some research problems.

2. Imbedding Types and Production Systems

It has been explained in detail elsewhere (esp. [GKP10, Gro14, GKMT18]) how the set of imbeddings for a graph can be partitioned, according to ***imbedding types*** (abbr. ***i-types***) that correspond to incidences of face-boundary walks at the root-vertices. We abbreviate "face-boundary walk" as ***fb-walk***. Our imbeddings are taken to be oriented.

The general idea is that an i-type of a graph with arbitrarily many root-vertices can be represented by a list of cyclic strings of root-vertices. To each fb-walk that is incident at one or more root-vertices, we associate the cyclic string of occurrences of root-vertices. The number of occurrences of each root-vertex in the string equals its number of occurrences on a complete traversal of the corresponding fb-walk.

An algorithmic process is given in [GKMT18] for constructing such a partition, and it is demonstrated that the number of i-types can be quite large, even for relatively small valences at the root-vertices. We are sometimes able to reduce the number of i-types by consideration of symmetries, for instance, when two or more root-vertices are in the same orbit under the action of the automorphism group.

When the front subset V of root-vertices for a linear family of graphs has a single vertex v_1 of valence 2, we can partition the imbeddings into the orientable surface S_i of each graph $(G_n, u_{n,1}, v_{n,1})$ in the H-linear sequence into two i-types:

- the subset of imbeddings into S_i such that two different face-boundary walks are incident at the root-vertex $v_{n,1}$, whose cardinality is denoted by $d_i(G_n)$;
- the subset of imbeddings into S_i such that the same face-boundary walk is twice incident at the root-vertex $v_{n,1}$, whose cardinality is denoted by $s_i(G_n)$.

If we labeled the single root-vertex of the front subset V as 0, the formal names of the i-types would be (0)(0) and (00). Here, we use the mnemonic designators d_i and s_i, as in many previous papers on genus polynomials, with $d_i(G) + s_i(G) = g_i(G)$.

The ***partial genus polynomials*** for G_n for this partitioning into imbedding types are

$$\begin{aligned}\Gamma^d_{G_n}(z) &= d_0(G_n) + d_1(G_n)z + d_2(G_n)z^2 + \cdots + d_{\gamma_{\max}(G)}(G)z^{\gamma_{\max}(G)} \text{ and}\\ \Gamma^s_{G_n}(z) &= s_0(G_n) + s_1(G_n)z + s_2(G_n)z^2 + \cdots + s_{\gamma_{\max}(G)}(G)z^{\gamma_{\max}(G)}.\end{aligned}$$

The column vector

$$\begin{bmatrix} d_0(G_n) + d_1(G_n)z + d_2(G_n)z^2 + \cdots + d_{\gamma_{\max}(G)}(G_n)z^{\gamma_{\max}(G)} \\ s_0(G_n) + s_1(G_n)z + s_2(G_n)z^2 + \cdots + s_{\gamma_{\max}(G)}(G_n)z^{\gamma_{\max}(G)} \end{bmatrix}$$

is called the ***pgd-vector***, where "pgd" stands for "partial-genus-distribution". We observe that the sum of the coordinates of the pgd-vector is the genus polynomial.

For instance, the pgd-vector for the graph G_0 of Figure 1.1 is

$$V_0(z) = \begin{bmatrix} 2 + 8z \\ 6z \end{bmatrix}. \tag{2.1}$$

To obtain the pgd-vectors for the iterated graphs after the initial graph of an H-linear sequence, we derive a ***production*** for each imbedding type. In this case, the production for imbedding type d is

$$d_i(G_n) \rightarrow 8d_i(G_{n+1}) + 48d_{i+1}(G_{n+1}) + 24s_{i+1}(G_{n+1}) + 16s_{i+2}(G_{n+1}).$$

This production means that starting from a fixed imbedding $G_n \to S_i$ of type d, the imbeddings of G_{n+1} obtainable by amalgamating the root $v_{n,1}$ of G_n to the root u_1 of H can be inventoried as follows, relative to the root-vertex $v_{n+1,1}$ of G_{n+1} (details of hand-calculation given by (4.3) and (4.4) of [GKP10]):

- 8 type-d imbeddings in S_i;
- 48 type-d imbeddings in S_{i+1};
- 24 type-s imbeddings in S_{i+1};
- 16 type-s imbeddings in S_{i+2}.

The sum of these counts is 96, which is equal to the product (i.e., 2^4) of the numbers of rotations at all the non-root vertices of H times the ratio (i.e., 6) of the number of rotations at the vertex $v_{n,1} = u_1$ of G_{n+1} to the number of rotations at $v_{n,1}$ in G_n. That sum is equal to the number of imbeddings of G_n obtainable by extension (using H) from a given embedding of G_n.

A "short form" of the production for type d is

$$d_i \;\rightarrow\; 8d_i + 48d_{i+1} + 24s_{i+1} + 16s_{i+2}. \tag{2.2}$$

We also used a program written by Imran Khan to calculate this production and others. Similarly, we have the following production for imbedding type s:

$$s_i \;\rightarrow\; 12d_i + 48d_{i+1} + 36s_{i+1}. \tag{2.3}$$

If we apply productions (2.2) and (2.3) to the pgd-vector (2.1) for G_0, we obtain the pgd-vector for G_1. A convenient way to apply a set of productions to a pgd-vector is to form a ***production matrix*** by representing each production as a column. For the K_4-linear sequence of Figure 1.1, we have the production matrix

$$M_{K_4}(z) \;=\; \begin{bmatrix} 8+48z & 12+48z \\ 24z+16z^2 & 36z \end{bmatrix}. \tag{2.4}$$

Letting $V_n(z)$ denote the pgd-vector of G_n, for $n = 0, 1, 2, \ldots$, we have

$$M_{K_4}(z)V_n(z) \;=\; V_{n+1}(z). \tag{2.5}$$

This approach can readily be generalized to arbitrarily many i-types. No matter how many i-types, the genus polynomial of the graph G_n is the sum of the coordinates of the pgd-vector $V_n(z)$. When the number of i-types equals k, each multiplication of the k-dimensional pgd-vector by the $k \times k$ production matrix requires k^2 multiplications of polynomials.

3. Linear Recursions for Genus Polynomials

In this section, we use the Cayley-Hamilton theorem to derive a method to calculate genus polynomials for linear families with k i-types, that requires only k multiplications of polynomials to proceed from $\Gamma_{G_n}(z)$ to $\Gamma_{G_{n+1}}(z)$.

Theorem 3.1 (Cayley-Hamilton Theorem)**.** *Let A be a $k \times k$ matrix over a commutative ring, with characteristic polynomial $\varphi(\lambda) = det(\lambda I - A)$. Then*

$$\varphi(A) \;=\; A^k + b_1A^{k-1} + \cdots + b_{k-1}A + b_kI \;=\; \boldsymbol{0}. \tag{3.1}$$

That is, every square matrix satisfies its own characteristic equation. □

Theorem 3.2. *Let k be the number of imbedding types for an H-linear sequence of graphs $\{G_n\}$. Then there is an ℓ^{th}-order linear recursion for the sequence of genus polynomials of these graphs, for some $\ell \le k$,*

$$\Gamma_{G_n}(z) \,=\, c_1(z)\Gamma_{G_{n-1}}(z) + c_2(z)\Gamma_{G_{n-2}}(z) + \cdots + c_\ell(z)\Gamma_{G_\ell}(z) \tag{3.2}$$

with coefficients $c_j(z)$ in the polynomial ring $\mathbb{Z}[z]$.

Proof. Suppose that the i-types of graph family $\{G_n\}$ are denoted $\tau^1, \tau^2, \ldots, \tau^k$. The set of embeddings of type τ^j of G_n into the surface S_i is denoted $\tau_i^j(G_n)$. In this notation, any of the k *partial genus polynomials* of $(G_n, v_{n,1}, v_{n,2}, \ldots, v_{n,s})$ may be written as

$$\Gamma^j_{G_n}(z) \,=\, \sum_{i\ge 0} |\tau_i^j(G_n)| z^i, \quad \text{for } 1 \le j \le k.$$

By face-tracing, as indicated by [GKP10], or by string operations, as described by [GKMT18], we can build a system of *productions* of the form

$$\tau^j(G_{n-1}) \;\to\; c_{j,1}(z)\tau^1(G_n) + c_{j,2}(z)\tau^2(G_n) + \cdots + c_{j,k}(z)\tau^k(G_n).$$

From this system of productions, we can induce a system of k simultaneous recursions of the following form:

$$\begin{aligned}
\Gamma^1_{G_n}(z) &= m_{11}(z)\Gamma^1_{G_{n-1}}(z) + m_{12}(z)\Gamma^2_{G_{n-1}}(z) + \cdots + m_{1k}(z)\Gamma^k_{G_{n-1}}(z)\\
\Gamma^2_{G_n}(z) &= m_{21}(z)\Gamma^1_{G_{n-1}}(z) + m_{22}(z)\Gamma^2_{G_{n-1}}(z) + \cdots + m_{2k}(z)\Gamma^k_{G_{n-1}}(z)\\
&\vdots\\
\Gamma^k_{G_n}(z) &= m_{k1}(z)\Gamma^1_{G_{n-1}}(z) + m_{k2}(z)\Gamma^2_{G_{n-1}}(z) + \cdots + m_{kk}(z)\Gamma^k_{G_{n-1}}(z)
\end{aligned} \tag{3.3}$$

where $m_{ij}(z) \in \mathbb{Z}[z]$ for $i, j = 1, 2, \ldots, k$.

The system (3.3) is representable in vector form as

$$V_{G_n}(z) \,=\, M(z) \cdot V_{G_{n-1}}(z). \tag{3.4}$$

where $V_{G_n}(z)$ is the pgd-vector and $M(z)$ is the production matrix, which generalizes (2.4). In an abstract combinatorial context (e.g., see [St86]), such a matrix has been called a *transfer matrix*.

Let us now suppose that the characteristic polynomial of the production matrix $M(z)$ is

$$\lambda^k + b_1(z)\lambda^{k-1} + \cdots + b_{k-1}(z)\lambda + b_k(z).$$

Then it follows from the Cayley-Hamilton theorem that

$$M(z)^k \,+\, b_1(z)M(z)^{k-1} \,+\, \cdots \,+\, b_{k-1}(z)M(z) \,+\, b_k(z) \,=\, 0.$$

Multiplying by the matrix $M(z)^{n-k-1}$, we obtain

$$M(z)^{n-1} + b_1(z)M(z)^{n-2} + \cdots + b_{k-1}(z)M(z)^{n-k} + b_k(z)M(z)^{n-k-1} \,=\, 0. \tag{3.5}$$

Applying Equation (3.5) to the column vector $V_{G_1}(z)$ yields

$$(3.6)\quad M(z)^{n-1}V_{G_1}(z) + b_1(z)M(z)^{n-2}V_{G_1}(z) + \cdots + b_{k-1}(z)M(z)^{n-k}V_{G_1}(z) + b_k(z)M(z)^{n-k-1}V_{G_1}(z) = 0,$$

and, in turn,

$$(3.7)\quad V_{G_n}(z) + b_1(z)V_{G_{n-1}}(z) + \cdots + b_{k-1}(z)V_{G_{n-k+1}}(z) + b_k(z)V_{G_{n-k}}(z) = 0.$$

Multiplying the preceding vector equation on the left by the k-dimensional row-vector $(1, 1, \ldots, 1)$, we obtain the genus polynomial equation

$$\Gamma_{G_n}(z) + b_1(z)\Gamma_{G_{n-1}}(z) + \cdots + b_{k-1}(z)\Gamma_{G_{n-k+1}}(z) + b_k(z)\Gamma_{G_{n-k}}(z) = 0,$$

which is readily transformed into a k^{th}-order homogeneous recursion of the form

$$(3.8)\quad \Gamma_{G_n}(z) = c_1(z)\Gamma_{G_{n-1}}(z) + c_2(z)\Gamma_{G_{n-2}}(z) + \cdots + c_k(z)\Gamma_{G_{n-k}}(z). \quad \square$$

As described on p302 of [Hog14], we can elaborate on Recurrence (3.8):

Theorem 3.3. *The coefficient $c_j(z)$ (associated with the production matrix $M_H(z)$) in Recurrences* (3.2) *and* (3.8) *is given by*

$$(3.9)\quad c_j(z) = (-1)^{j+1} \sum \textit{(all } j \times j \textit{ principal minors of } M\textit{)}. \quad \square$$

When there are only two imbedding types, it follows from Theorem 3.2 that the recursion takes the form

$$(3.10)\quad \Gamma_{G_n}(z) = c_1(z)\Gamma_{G_{n-1}}(z) + c_2(z)\Gamma_{G_{n-2}}(z).$$

It follows from Theorem 3.3 that the coefficients in Recurrence (3.10) are

$$\begin{aligned} c_1(z) &= trace(M_H(z)) \text{ and} \\ c_2(z) &= -det(M_H(z)). \end{aligned}$$

Recalling from (2.4) that the production matrix

$$(3.11)\quad M_{K_4}(z) = \begin{bmatrix} 48z + 8 & 48z + 12 \\ 16z^2 + 24z & 36z \end{bmatrix}.$$

has $trace(M_{K_4}(z)) = 84z + 8$ and $det(M_{K_4}(z)) = -768z^3 + 384z^2$, we can now confirm the correctness of the recurrence system:

$$\begin{aligned} \Gamma_{G_n}(z) &= (84z + 8)\Gamma_{G_{n-1}}(z) + (768z^3 - 384z^2)\Gamma_{G_{n-2}}(z) \\ \Gamma_{G_0}(z) &= 14z + 2 \\ \Gamma_{G_1}(z) &= 128z^3 + 1112z^2 + 280z + 16, \end{aligned}$$

which we previously gave as Recurrence (1.1).

3.1. **Another way to calculate the recurrence.** It is possible to calculate the recurrence for the genus polynomials of a linear sequence of graphs $\{G_n\}$ without calculating the production matrix. Let's suppose that there are k imbeddings types, and that we somehow know, e.g., by ad hoc methods, the first $2k$ genus polynomials. Then the recurrence

$$\Gamma_{G_n}(z) \;=\; c_1(z)\Gamma_{G_{n-1}}(z) \;+\; c_2(z)\Gamma_{G_{n-2}}(z) \;+\; \cdots \;+\; c_k(z)\Gamma_{G_{n-k}}(z) \tag{3.12}$$

leads to a system of k simultaneous linear equations with k unknowns, which are the coefficients $c_1(z), c_2(z), \ldots, c_k(z)$. The need for $2k$ genus polynomials is illustrated by Example 3.1.

Example 3.1. For instance, suppose that (H, u, v) is the graph $(K_4 - e, u, v)$, in which the two 2-valent vertices are the roots, as illustrated in Figure 3.1.

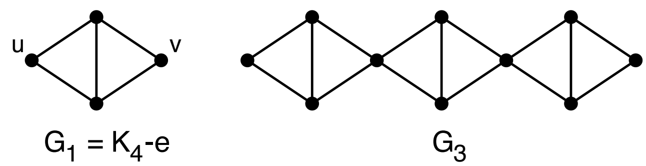

FIGURE 3.1. Sequence of $(K_4 - e)$-chains.

As in Example 1.1, there are two i-types for each of the graphs (G_n, v), and we have the recurrence

$$\Gamma_{G_n}(z) \;=\; c_1(z)\Gamma_{G_{n-1}}(z) \;+\; c_2(z)\Gamma_{G_{n-2}}(z)$$

with initial values (calculated by summing pgd-vectors)

$$\begin{aligned}
\Gamma_{G_1}(z) &= 2z + 2\\
\Gamma_{G_2}(z) &= 24z^2 + 56z + 16\\
\Gamma_{G_3}(z) &= 288z^3 + 1120z^2 + 768z + 128\\
\Gamma_{G_4}(z) &= 3456z^4 + 19328z^3 + 22784z^2 + 8704z + 1024.
\end{aligned}$$

This leads us to the simultaneous equations

$$\begin{aligned}
\Gamma_{G_3}(z) &= c_1(z)\Gamma_{G_2}(z) + c_2(z)\Gamma_{G_1}(z)\\
\Gamma_{G_4}(z) &= c_1(z)\Gamma_{G_3}(z) + c_2(z)\Gamma_{G_2}(z),
\end{aligned}$$

which we instantiate as

$$\begin{aligned}
288z^3 + 1120z^2 + 768z + 128 &= c_1(z)(24z^2 + 56z + 16) + c_2(z)(2z + 2)\\
3456z^4 + 19328z^3 + {}&\\
22784z^2 + 8704z + 1024 &= c_1(z)(288z^3 + 1120z^2 + 768z + 128)\\
&\qquad + c_2(z)(16 + 56z + 24z^2),
\end{aligned}$$

with the solutions

$$\begin{aligned} c_1(z) &= 20z+8 \\ c_2(z) &= -96z^2. \end{aligned}$$

Thus, our recurrence is

$$\Gamma_{G_n}(z) \;=\; (20z+8)\Gamma_{G_{n-1}}(z) \;+\; (-96z^2)\Gamma_{G_{n-2}}(z).$$

Example 3.2. We next examine a $\dot{D}_3$-linear family, illustrated in Figure 3.2.

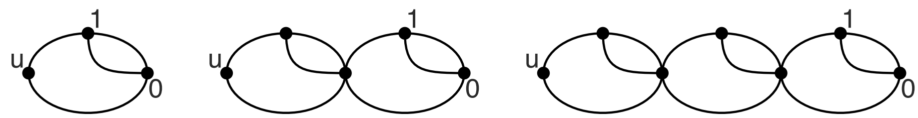

FIGURE 3.2. Chains of copies of $\dot{D}_3$.

Although the number of i-types needed is not obvious, we may discover that any production matrix for this linear family would have rank = 2, which implies that various i-types could be combined, as described in [GKMT18]. We calculate the first four genus polynomials.

$$\begin{aligned} \Gamma_1(z) &= 2z+2 \\ \Gamma_2(z) &= 48z^2+120z+24 \\ \Gamma_3(z) &= 1152z^3+4992z^2+2784z+288 \\ \Gamma_4(z) &= 27648z^4+176640z^3+185088z^2+49536z+3456. \end{aligned}$$

Solving for the unknown coefficients $c_1(z)$ and $c_2(z)$ as in the previous example, we obtain the recurrence

$$\Gamma_n(z) \;=\; (40z+12)\Gamma_{n-1}(z)+(-384z^2+192z)\Gamma_{n-2}(z).$$

4. (H,u,v)-Linear Sequences of Recurrence Type (p,q)

This section and the next section focus on H-linear sequences of graphs for which the minimum number of imbedding types is two, and accordingly, the associated linear recurrences are of order two. In general, when the he genus polynomial $\Gamma_{G_n}(z)$ are polynomials of degrees of the coefficients of the immediate k predecessors $\Gamma_{G_{n-1}}(z), \Gamma_{G_{n-2}}(z), \ldots, \Gamma_{G_{n-k}}(z)$ of the genus polynomial $\Gamma_{G_n}(z)$ are, respectively,

$$d_1, d_2, \ldots, d_k,$$

we say that the recurrence and the associated H-linear sequence are of ***type*** $(d_1, d_2, \ldots, d_k)$. When $k=2$, we often give the type as (p,q).

Proofs of log-concavity of the genus polynomials for linear graph families (e.g.,[FGS89, GKP14, GMT14, Stah91, Stah97]) have mostly been based on the specific recurrences for those genus polynomials. A broader approach exemplified by [GMTW16a, GMTW16b], proves that under highly general algebraic conditions, a recurrence for polynomials (with coefficients in $\mathbb{Z}[z]$, of type (0,1) or type (1,0)) yields a sequence of real-rooted polynomials. For instance, the recursions for the genus polynomials ladders and for the cobblestone paths are of type (0,1) and satisfy the algebraic conditions.

A later concern in this section and the next is recurrences of type (1,1) that result, respectively, from these two particular circumstances:

(1) Each of the two subsets U and V of the bipartition of the root-vertices has a single 2-valent vertex, u_1 and v_1, respectively.
(2) The subset U has two adjacent 1-valent vertices u_1 and u_2, and the subset V has two 2-valent vertices v_1 and v_2.

We will prove that for two infinite classes of linear families described in §4.1 and §4.2 that the associated genus polynomials are log-concave.

Theorem 4.1. *Let (H, u, v) be a 2-connected graph whose root-vertices u and v are 2-valent. Then an H-linear sequence is of type (p, q), with*

$$(a)\ \gamma_{\max}(H) \leq p \leq \gamma_{\max}(H) + 1 \quad \textit{and} \quad (b)\ q \leq 2\gamma_{\max}(H) + 1.$$

Proof. In string notation, using 0 for u and 1 for v, the ten possible i-types for (H, u, v) are

$$(0)(0)(1)(1), (0)(01)(1), (01)(01), (0)(0)(11), (0)(011),$$
$$(00)(1)(1), (001)(1), (00)(11), (0011), (0101).$$

In the notation of [GKP10], which offers some convenience here, they would be written as

$$dd^0, dd', dd'', ds^0, ds', sd^0, sd', ss^0, ss^1, \text{ and } ss^2,$$

respectively. Accordingly, the genus polynomial for (H, u, v) can be partitioned into ten partial genus polynomials. In Table 4.1, we give a condensed set of production rules for vertex-amalgamating the single-rooted graph $(G_{n-1}, v_{n-1,1})$ at the first root of the double-rooted graph $(H_n, u_{n,1}, v_{n,1})$, adapted from Table 2 of [GKP10]. A superscript bullet ($\bullet$) is used when all variations of the mainscript have the same consequence. For instance $ss_j^\bullet$ means that the rule applies to ss_j^0, ss_j^1, and ss_j^2.

By applying the rules of Table 4.1, we obtain the production matrix

$$M(z) = \begin{bmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{bmatrix}$$

TABLE 4.1. Productions for amalgamation $(G_{n-1}, v_{n-1,1}) * (H_n, u_{n,1}, v_{n,1})$.

production
$d_i(G_{n-1}) * dd^0_j(H_n) \;\rightarrow\; 4d_{i+j}(G_n) + 2d_{i+j+1}(G_n)$
$d_i(G_{n-1}) * dd'_j(H_n) \;\rightarrow\; 4d_{i+j}(G_n) + 2d_{i+j+1}(G_n)$
$d_i(G_{n-1}) * dd''_j(H_n) \;\rightarrow\; 4d_{i+j}(G_n) + 2s_{i+j+1}(G_n)$
$s_i(G_{n-1}) * dd^\bullet_j(H_n) \;\rightarrow\; 6d_{i+j}(G_n)$
$d_i(G_{n-1}) * ds^\bullet_j(H_n) \;\rightarrow\; 4s_{i+j}(G_n) + 2s_{i+j+1}(G_n)$
$s_i(G_{n-1}) * ds^\bullet_j(H_n) \;\rightarrow\; 6s_{i+j}(G_n)$
$d_i(G_{n-1}) * sd^\bullet_j(H_n) \;\rightarrow\; 6d_{i+j}(G_n)$
$s_i(G_{n-1}) * sd^\bullet_j(H_n) \;\rightarrow\; 6d_{i+j}(G_n)$
$d_i(G_{n-1}) * ss^0_j(H_n) \;\rightarrow\; 6s_{i+j}(G_n)$
$d_i(G_{n-1}) * ss^1_j(H_n) \;\rightarrow\; 6s_{i+j}(G_n)$
$d_i(G_{n-1}) * ss^2_j(H_n) \;\rightarrow\; 4s_{i+j}(G_n) + 2d_{i+j}(G_n)$
$s_i(G_{n-1}) * ss^\bullet_j(H_n) \;\rightarrow\; 6s_{i+j}(G_n)$

for an H-linear sequence, where

$$\begin{aligned}
m_{11}(z) &= (4+2z)(dd^0 + dd') + 4dd'' + 6(sd^0 + sd') + 2ss^2 \\
m_{12}(z) &= 6(dd^0 + dd' + dd'' + sd^0 + sd') \\
m_{21}(z) &= 2zdd'' + (4+2z)(ds^0 + ds') + 6(ss^0 + ss^1) + 4ss^2 \\
m_{22}(z) &= 6(ds^0 + ds' + ss^0 + ss^1 + ss^2).
\end{aligned}$$

Each of the imbedding types for a graph corresponds to a polynomial invariant of that graph in the indeterminate z; however, we have written dd'' instead of $dd''_H(z)$, and so on, in order to fit each of the matrix entries onto a single line.

We can calculate that the trace of the production matrix $M(z)$ equals

$$(4+2z)(dd^0 + dd') + 4dd'' + 6(ds^0 + ds' + sd^0 + sd' + ss^0 + ss^1) + 8ss^2. \tag{4.1}$$

We observe that the degree of $\gamma_{\max}(H)$ equals the maximum of the degrees of the partial genus polynomials

$$dd^0, dd', dd'', ds^0, ds', sd^0, sd', ss^0, ss^1, ss^2,$$

since the genus polynomial $\Gamma_H(z)$ is the sum of those partial genus polynomials. Moreover, since our Formula (4.1) for the trace of $M(z)$ equals the genus polynomial plus a sum of polynomials all of whose coefficients are non-negative, it follows that the degree p of the trace of $M(z)$ is at least as large as $\gamma_{\max}(H)$. We see also that the upper bound $\gamma_{\max}(H) + 1$ for p is achieved only when the degree of $dd^0_H(z) + dd'_H(z)$ equals $\gamma_{\max}(H)$. This completes the proof of Inequality (a).

We observe further that the degree of each of the terms of $m_{11}(z)$ is at most one more than the degree of $\Gamma_H(z)$ and that the degree of each of the terms of $m_{22}(z)$ is at most the degree of $\Gamma_H(z)$. It follows that the degree of $m_{11}(z)m_{22}(z)$ is at most $2\gamma_{\max}(H)+1$. Similarly, the degree of $m_{12}(z)m_{21}(z)$ is at most $2\gamma_{\max}(H)+1$. Thus,

$$\begin{aligned} q &= deg(det(M(z))) \\ &= deg\left(m_{11}(z)m_{22}(z) - m_{12}(z)m_{21}(z)\right) \\ &\leq max\left\{deg(m_{11}(z)m_{22}(z)),\, deg(m_{12}(z)m_{21}(z))\right\} \\ &\leq 2\gamma_{\max}(H)+1. \end{aligned}$$

□

4.1. **Closed necklaces.** In this subsection, we seek to characterize the graphs (H,u,v), where u and v are both 2-valent, such that an H-linear sequence is of type (1,1). By Theorem 4.1, we know that $\gamma_{\max}(H) \leq 1$. According to Theorem 1 of [CG93], a 2-connected graph H with $\gamma_{\max}(H)=1$ must either be a member of an infinite family called *necklaces*, illustrated in Figure 4.1 (and defined below) or one of the five sporadic graphs shown in Figure 4.2. (The name “PM” was chosen because the graph PM looks like a pac-man.)

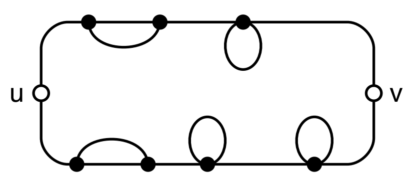

FIGURE 4.1. A double-rooted closed (2,3)-necklace.

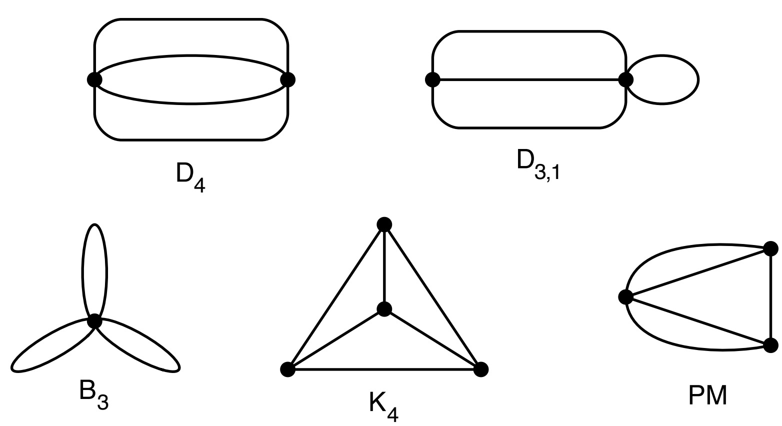

FIGURE 4.2. The five sporadic graphs with maximum genus 1.

A ***closed*** (r,s)***-necklace***, first defined by [GKR93], is formed from a cycle graph by attaching a loop at each of s vertices and doubling r of the cycle edges, so that none of the resulting vertices has valence larger than 4; here we designate two of the 2-valent vertices as root-vertices, as shown in Figure 4.1.

Theorem 4.2. *Let N be a closed (r,s)-necklace. Then the production matrix and initial pgd-vector for the sequence of N-linear graphs are*

$$M_{r,s}(z) = \begin{bmatrix} 2^r4^{s+1} & 6\cdot 2^r4^s \\ (4^r6^{s+1}-2^r4^{s+1})z & (4^r6^{s+1}-6\cdot 2^r4^s)z \end{bmatrix} \text{ and } V_0 = \begin{bmatrix} 2^r4^s \\ (4^r6^s - 2^r4^s)z \end{bmatrix}.$$

This corresponds to the type-(1,1) recurrence

$$\begin{aligned} \Gamma_{N_n}(z) &= trace(M_{r,s})\Gamma_{N_{n-1}}(z) - det(M_{r,s})\Gamma_{N_{n-2}}(z) \\ \Gamma_{N_1}(z) &= 2^r4^s + (4^r6^s - 2^r4^s)z \\ \Gamma_{N_2}(z) &= 4^{r+2s+1} + \left(6\cdot 2^r4^s(4^r6^s-2^r4^s) + 2^r4^s(4^r6^{s+1}-2^r4^{s+1})\right)z \\ &\quad + (4^r6^{s+1} - 6\cdot 2^r4^s)(4^r6^s-2^r4^s)z^2, \end{aligned} \tag{4.2}$$

where

$$trace(M_{r,s}) = 2^r4^{s+1} + (4^r6^{s+1} - 6\cdot 2^r4^s)z \quad \text{and} \tag{4.3}$$

$$\begin{aligned} det(M_{r,s}) &= 2^r4^{s+1}\left(4^r6^{s+1} - 6\cdot 2^r4^s\right)z \\ &\quad -6\cdot 2^r4^s(4^r6^{s+1}-2^r4^{s+1})z \\ &= -12\cdot 8^r\cdot 24^s z. \end{aligned} \tag{4.4}$$

Proof. Let (N_n, v_n) denote the chain of n copies of the (r,s)-necklace N. We observe that the number of imbeddings $N \to S_0$ is 2^r4^s. Since the total number of imbeddings of N is 4^r6^s, the initial pgd-vector is

$$V_0 = \begin{bmatrix} 2^r4^s \\ (4^r6^s - 2^r4^s)z \end{bmatrix}.$$

Using ad hoc methods, such as drawing pictures of the face-boundary walks incident at the root vertices and the effect of amalgamation on the number of such walks, one can show that an amalgamation of an imbedding of N to a given type-d imbedding of N_n, thereby forming an imbedding of N_{n+1}, has imbedding type d if and only if the given imbedding of N is in S_0 and the two edge-ends incident at root-vertex u of N are adjacent in the rotation at the amalgamated vertex $v_{n,1} = u$ in the imbedding of N_{n+1}. Moreover, the genus of the imbedding surface rises (in which case it rises by 1) if and only if the resulting imbedding of N_{n+1} is type s. Thus, there are $m_{11}(z) = 2^r4^{s+1}$ ways to amalgamate an imbedding of N to a given type-d imbedding of N_n so as to obtain a type-d imbedding. Since the total number of ways to amalgamate an imbedding of N to a given type-d imbedding of N_n is 4^r6^{s+1}, we can infer that the number of ways to obtain a type-s imbedding is $m_{21}(z) = (4^r6^{s+1} - 2^r4^{s+1})z$.

Continuing with ad hoc methods, one can show that an amalgamation of an imbedding of N to a given type-s imbedding of N_n has imbedding type d if and only if the given imbedding of N is in S_0. Moreover, the genus of the imbedding surface rises (in which case it rises by 1) if and only if the resulting imbedding of N_{n+1} is type s. Thus, there are $m_{12}(z) = 2^r 4^s$ ways to amalgamate an imbedding of N to a given imbedding of N_n so as to obtain a type-d imbedding. Since the total number of ways to amalgamate an imbedding of N to a given imbedding of N_n is $4^r 6^{s+1}$, we can infer that the number of ways to obtain a type-s imbedding is $m_{22}(z) = (4^r 6^{s+1} - 2^r 4^s)z$. □

Theorem 4.3. *The polynomial $\Gamma_{N_n}(z)$ is real-rooted for all $n \geq 0$.*

Proof. The leading coefficient of every genus polynomial is positive, so $\{\Gamma_{N_n}(z)\}$ is a sequence of standard polynomials. In Recurrence 4.2 the coefficients of $\Gamma_{N_{n-1}}(z)$ and $\Gamma_{N_{n-2}}(z)$ are real. Furthermore. we can prove inductively that $\deg(\Gamma_{N_n}(z)) = \deg(\Gamma_{N_{n-1}}(z))+1$. Moreover, the coefficients of $\Gamma_{N_n}(z)$ are non-negative, from the definition of a genus polynomial. By Theorem 4.2, we have that $-\det(M_{r,s}(z)) = 12 \cdot 8^r \cdot 24^s z$, which is non-positive for all $z \leq 0$. Thus, by Corollary 2.4 of [LW07], the polynomials $\{\Gamma_{N_n}(z)\}$ are real-rooted. □

Corollary 4.4. *The polynomial $\Gamma_{N_n}(z)$ is log-concave for all $n \geq 0$.*

Proof. It follows by Newton's real-roots theorem that every real-rooted polynomial is log-concave. □

We recall that the n^{th} ***Chebyshev polynomial*** $U_n(x)$ ***of the second kind*** is defined as follows:

$$U_n(x) \;=\; 2xU_{n-1}(x) - U_{n-2}(x),$$

with the initial values $U_0(x) = 1$ and $U_1(x) = 2x$. Moreover, the explicit formula for $U_n(x)$ is given by

$$U_n(x) \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k}. \tag{4.5}$$

Theorem 4.5. *Let $i^2 = -1$, $d = 2\sqrt{3}\sqrt{8}^r\sqrt{24}^s\sqrt{z}$ and $t = 2^r 2^s(2^{s+2}+6(2^r 3^s - 2^s)z)$. Then*

$$\Gamma_{N_n}(z) = (id)^{n-1}\left[U_{n-1}\left(\frac{t}{2id}\right)\Gamma_{N_1}(z) - idU_{n-2}\left(\frac{t}{2id}\right)\Gamma_{N_0}(z)\right],$$

where

$$\begin{aligned}
\Gamma_{N_0}(z) &= 2^r 4^s + (4^r 6^s - 2^r 4^s)z \\
\Gamma_{N_1}(z) &= 4^{r+2s+1} + \big(6 \cdot 2^r 4^s (4^r 6^s - 2^r 4^s) + 2^r 4^s (4^r 6^{s+1} - 2^r 4^{s+1})\big)\, z \\
&\quad + 6(4^r 6^s - 2^r 4^s)^2 z^2,
\end{aligned}$$

and U_m is the m^{th} Chebyshev polynomial of the second kind.

Proof. By Theorem 4.2, we have that

$$\frac{\Gamma_{N_n}(z)}{(id)^n} \;=\; \frac{t}{id}\frac{\Gamma_{N_{n-1}}(z)}{(id)^{n-1}} - \frac{\Gamma_{N_{n-2}}(z)}{(id)^{n-2}}, \quad n \geq 2.$$

Hence, by induction on n, we have

$$\Gamma_{N_n}(z) = (id)^{n-1}\left(U_{n-1}(t/(2id))\Gamma_{N_1}(z) - idU_{n-2}(t/(2id))\Gamma_{N_0}(z)\right),$$

as required. □

Theorem 4.6. *Let G be any of the five sporadic graphs of Figure 4.2, and let root-vertices u and v be inserted as subdivision points of any edge or pair of edges. Then the degree of the determinant of the production matrix of the (G,u,v)-linear sequence is at least 2.*

Proof. This is demonstrated by checking cases, preferably with a computational aid. □

Thus, whereas Theorem 4.2 implies that all the recurrences of closed (r,s) necklaces are of type $(1,1)$, Theorem 4.6 establishes that none of the possible H-linear sequences where H is any of the five sporadic graphs corresponds to a recurrence of type $(1,1)$.

4.2. **A recurrence of type** $(1,2)$**.** In this subsection, we consider the B_2-linear family of graphs illustrated in Figure 4.3, where B_2 denotes the bouquet with two loops. Its induced recurrence is of type $(1,2)$. We prove that all of its genus polynomials are real-rooted.

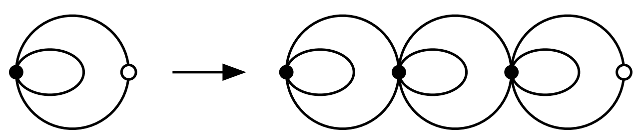

FIGURE 4.3. A B_2-linear family of graphs.

The B_2-chain of Figure 4.3 has the production matrix

$$M_{B_2}(z) = \begin{bmatrix} 20z+40 & 28z+56 \\ 60z & 36z \end{bmatrix}, \tag{4.6}$$

which induces the recursion

$$\Gamma_{G_n}(z) = (56z+40)\Gamma_{G_{n-1}}(z) + (960z^2+1920z)\Gamma_{G_{n-1}}(z) \tag{4.7}$$
$$\Gamma_{G_0}(z) = 2z+4 \tag{4.8}$$
$$\Gamma_{G_1}(z) = 160z^2+400z+160. \tag{4.9}$$

Since $\Gamma_{G_0}(-2) = \Gamma_{G_1}(-2) = 0$, it follows from the recursion that $\Gamma_{G_n}(z) = 0$ when $z = -2$, for all $n \geq 0$.

Theorem 4.7. *All the genus polynomials for the B_2-linear family of Figure 4.3 are real-rooted.*

Proof. By a simple induction on n, we can show that $\deg(\Gamma_{G_n}(z)) = n+1$ and that $\Gamma_{G_n}(-2) = 0$. Another straightforward induction argument shows that $(-1)^{n-1}\Gamma_{G_n}(z) > 0$, for all $n \geq 0$ and for any $z < -2$. Noticing that $960z^2 + 1920z < 0$ for all $-2 < z < 0$, we infer from Corollary 2.4 of [LW07] that the polynomials $\{\Gamma_{G_n}(z)\}$ are real-rooted for all $n \geq 0$. □

5. $(H, \{u_1, u_2\}, \{v_1, v_2\})$-Linear Sequences of Recurrence Type (p, q)

In this section, the rear root-vertices u_1 and u_2 of $(H, \{u_1, u_2\}, \{v_1, v_2\})$ are 1-valent and the front root-vertices v_1 and v_2 are 2-valent. We shall see how the corresponding H-linear sequence can be of type (1,1). Our exposition proceeds largely in parallel with Section 4.

Theorem 5.1. *Let $(H, \{u_1, u_2\}, \{v_1, v_2\})$ be a 2-connected graph with 1-valent rear root-vertices and adjacent, 2-valent front root-vertices. Then an H-linear sequence is of type (p, q), with*

$$(a)\ \gamma_{\max}(H) \leq p \leq \gamma_{\max}(H) + 1 \quad \textit{and} \quad (b)\ q \leq 2\gamma_{\max}(H) + 1.$$

Proof. We observe that the graph obtained by amalgamating the 1-valent vertices u_1 and u_2 to adjacent 2-valent vertices v_1 and v_2, respectively of some previous copy of H is exactly the same as the graph that would be obtained by first joining u_1 and u_2 with a new edge and then performing an edge-amalgamation of edge u_1u_2 with edge v_1v_2. Accordingly, we can use the ten imbedding types described in [PKG10] for a doubly edge-rooted graph such that each endpoint of each root-edge is 2-valent.

$$dd^0, dd', dd'', ds^0, ds', sd^0, sd', ss^0, ss^1, \text{ and } ss^2.$$

We see that the names of these imbedding types are the same as for two 2-valent vertex roots. However, the corresponding productions, adapted from [PKG10], are different from those in Table 4.1.

By applying the rules of Table 5.1, we obtain the production matrix

$$M(z) = \begin{bmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{bmatrix}$$

TABLE 5.1. Productions for amalgamating the graph $(G_{n-1}, v_{n-1,1}, v_{n-1,2})$ with the graph $(H, \{u_1, u_2\}, \{v_1, v_2\})$.

production		
$d_i(G_{n-1}) * dd_j^0(H_n)$	$\rightarrow$	$2d_{i+j}(G_n) + 2d_{i+j+1}(G_n)$
$s_i(G_{n-1}) * dd_j^0(H_n)$	$\rightarrow$	$4d_{i+j}(G_n)$
$d_i(G_{n-1}) * dd_j'(H_n)$	$\rightarrow$	$2d_{i+j}(G_n) + 2d_{i+j+1}(G_n)$
$s_i(G_{n-1}) * dd_j^1(H_n)$	$\rightarrow$	$4d_{i+j}(G_n)$
$d_i(G_{n-1}) * dd_j''(H_n)$	$\rightarrow$	$2d_{i+j}(G_n) + 2s_{i+j+1}(G_n)$
$s_i(G_{n-1}) * dd_j''(H_n)$	$\rightarrow$	$4d_{i+j}(G_n)$
$d_i(G_{n-1}) * ds_j^\bullet(H_n)$	$\rightarrow$	$2s_{i+j}(G_n) + 2s_{i+j+1}(G_n)$
$s_i(G_{n-1}) * ds_j^\bullet(H_n)$	$\rightarrow$	$4s_{i+j}(G_n)$
$d_i(G_{n-1}) * sd_j^\bullet(H_n)$	$\rightarrow$	$4d_{i+j}(G_n)$
$s_i(G_{n-1}) * sd_j^\bullet(H_n)$	$\rightarrow$	$4d_{i+j}(G_n)$
$d_i(G_{n-1}) * ss_j^0(H_n)$	$\rightarrow$	$4s_{i+j}(G_n)$
$d_i(G_{n-1}) * ss_j^1(H_n)$	$\rightarrow$	$4s_{i+j}(G_n)$
$d_i(G_{n-1}) * ss_j^2(H_n)$	$\rightarrow$	$2d_{i+j}(G_n) + 2s_{i+j}(G_n)$
$s_i(G_{n-1}) * ss_j^\bullet(H_n)$	$\rightarrow$	$4s_{i+j}(G_n)$

for an H-linear sequence, where

$$\begin{aligned}
m_{11}(z) &= (2+2z)(dd^0 + dd') + 2dd'' + 4(sd^0 + sd') + 2ss^2 \\
m_{12}(z) &= 4(dd^0 + dd' + dd'' + sd^0 + sd') \\
m_{21}(z) &= 2zdd'' + (2+2z)(ds^0 + ds') + 4(ss^0 + ss^1) + 2ss^2 \\
m_{22}(z) &= 4(ds^0 + ds' + ss^0 + ss^1 + ss^2).
\end{aligned}$$

As before, each of the imbedding types for a doubly rooted graph with two 2-valent roots is representable as a polynomial-valued invariant of the graph H in the indeterminate z.

We can calculate that the trace of the production matrix $M(z)$ equals

$$(5.1)\quad (2+2z)(dd^0 + dd') + 2dd'' + 4(ds^0 + ds' + sd^0 + sd' + ss^0 + ss^1) + 6ss^2.$$

We observe here again that the degree of $\gamma_{\max}(H)$ equals the maximum of the degrees of the partial genus polynomials

$$dd^0, dd', dd'', ds^0, ds', sd^0, sd', ss^0, ss^1, ss^2,$$

since the genus polynomial $\Gamma_H(z)$ is the sum of those partial genus polynomials. Moreover, since our Formula (5.1) for the trace of $M(z)$ equals the genus polynomial plus a sum of polynomials all of whose coefficients are non-negative, it follows that the degree p of the trace of $M(z)$ is at least as large as $\gamma_{\max}(H)$.

We again see also that the upper bound $\gamma_{\max}(H)+1$ for p is achieved only when the degree of $dd^0_H(z)+dd'_H(z)$ equals $\gamma_{\max}(H)$.

The proof of Inequality (b) is similar to the proof of the analogous inequality in Theorem 4.1. □

The ***open*** (r,s)-***necklace*** $(N, \{u_1,u_2\}, \{v_1,v_2\})$ can be formed from a closed (r,s)-necklace (N,u,v) by splitting (i.e., reverse contraction) the front root v into two adjacent 2-valent roots v_1 and v_2 and also splitting the rear root u into two adjacent 2-valent roots u_1 and u_2, followed by deletion of the edge joining u_1 and u_2. Figure 5.1 illustrates the open necklace formed from the closed necklace of Figure 4.1.

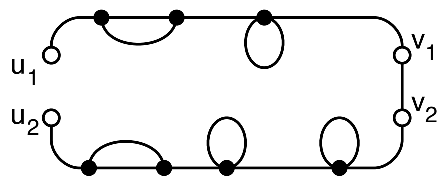

FIGURE 5.1. An open (2,3)-necklace.

Theorem 5.2. *Let N be an open (r,s)-necklace. Then the production matrix and initial pgd-vector for the sequence of N-linear graphs are*

$$M_{r,s}(z) = \begin{bmatrix} 2^{r+1}4^s & 2^r4^{s+1} \\ (4^{r+1}6^s - 2^{r+1}4^s)z & (4^{r+1}6^s - 2^r4^{s+1})z \end{bmatrix} \text{ and } V_0 = \begin{bmatrix} 2^r4^s \\ (4^r6^s - 2^r4^s)z \end{bmatrix}.$$

This corresponds to the type-(1,1) recurrence

$$\begin{aligned} \Gamma_{N_n}(z) &= trace(M_{r,s})\Gamma_{N_{n-1}}(z) - det(M_{r,s})\Gamma_{N_{n-2}}(z) \\ \Gamma_{N_0}(z) &= 2^r4^s + (4^r6^s - 2^r4^s)z \\ \Gamma_{N_1}(z) &= 2\cdot 4^{r+2s} + \left(2^r4^{s+1}(4^r6^s - 2^r4^s) + (4^r6^{s+1} - 2^{r+1}4^s)2^r4^s\right) z \\ &\quad + (4^r6^{s+1} - 2^r4^{s+1})(4^r6^s - 2^r4^s)z^2. \end{aligned}$$

Proof. Let (N_n, v_n) denote the chain of n copies of the open (r,s)-necklace N. Let $\overline{N}$ be the closed necklace obtained from N by joining the root-vertices u and v. The initial graph for the N-linear sequence is the same as for $\overline{N}$. Thus, the pgd-vector for N is the same as for $\overline{N}$, i.e.,

$$V_0 = \begin{bmatrix} 2^r4^s \\ (4^r6^s - 2^r4^s)z \end{bmatrix}.$$

As in the proof of Theorem 4.2, drawing pictures as in [PKG10] (or other ad hoc method) can yield the entries of the production matrix $M_{r,s}(z)$. □

Theorem 5.3. *Let (N, U, V) be an open (r, s)-necklace and let $\{N_n\}$ be the sequence of N-linear graphs. Then the polynomial $\Gamma_{N_n}(z)$ is real-rooted for all $n \geq 0$.*

Proof. By Theorem 5.2, we know that $-\det(M_{r,s}(z)) = 8^{r+1} \cdot 24^s z$, which is non-positive for all $z \leq 0$. It follows from Corollary 2.4 of [LW07], as in the proof of Theorem 4.3, that the polynomials $\{\Gamma_{N_n}(z\})$ are real-rooted. □

Corollary 5.4. *The polynomial $\Gamma_{N_n}(z)$ is log-concave for all $n \geq 0$.*

Proof. This follows by Newton's theorem that every real-rooted polynomial is log-concave. □

Theorem 5.5. *Let $i^2 = -1$, $d = \sqrt{8}^{r+1}\sqrt{24}^{s}\sqrt{z}$ and $t = 2^{r+1}2^{s+1}(2^{s-1} + (2^r 3^s - 2^s)z)$. Then*

$$\Gamma_{N_n}(z) = (id)^{n-1}\left[U_{n-1}\left(\frac{t}{2id}\right)\Gamma_{N_1}(z) - idU_{n-2}\left(\frac{t}{2id}\right)\Gamma_{N_0}(z)\right],$$

where $\Gamma_{N_0}(z)$ and $\Gamma_{N_1}(z)$ are given in Theorem 5.2, and U_m is the m^{th} Chebyshev polynomial of the second kind.

Proof. By Theorem 5.2, we have that

$$\frac{\Gamma_{N_n}(z)}{(id)^n} = \frac{t}{id}\frac{\Gamma_{N_{n-1}}(z)}{(id)^{n-1}} - \frac{\Gamma_{N_{n-2}}(z)}{(id)^{n-2}}, \quad n \geq 2.$$

Hence, by induction on n, we have

$$\Gamma_{N_n}(z) = (id)^{n-1}\left(U_{n-1}(t/(2id))\Gamma_{N_1}(z) - idU_{n-2}(t/(2id))\Gamma_{N_0}(z)\right),$$

as required. □

Theorem 5.6. *Let G be any of the five sporadic graphs of Figure 4.2, and let root-vertex pairs u_1, u_2 and v_1, v_2 be inserted as subdivision points of any edge or pair of edges, such that u_1 and u_2 are adjacent and also v_1 and v_2 are adjacent. Then the degree of the determinant of the production matrix of the (G, u, v)-linear sequence is at least 2.*

Proof. Once again, we can check cases with a computational aid. □

Analogous to Theorem 4.6, Theorem 5.6 excludes another possible way that any of the five sporadic graphs might H-linear sequences where H is corresponds to a recurrence of type $(1, 1)$.

6. Upper-Imbeddable Iterated Graphs

A graph G is said to be ***upper-embeddable*** if $\gamma_{\max}(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G)$ denotes the cycle rank of G.

When the iterated graph (H, u, v) is upper-embeddable, we can strengthen Theorem 4.1, by dropping the limitations on valences of the roots, and by giving exact values for the maximum genera. It follows from a theorem of [Kun74] that almost all graphs are upper-imbeddable. The bounds obtained for the coefficients in the genus polynomial recursion depend on the parity of the cycle rank of H.

Theorem 6.1. *Let (H, u, v) be an upper-imbeddable graph, in which the cycle rank $\beta(H)$ is even. Then the maximum genus of each graph G_n in the corresponding H-linear sequence $\{(G_n, v_n), n \geq 1\}$ is $n\gamma_{\max}(H)$.*

Proof. Since the cycle rank of H is even, it follows that $|V_H| - |E_H|$ is odd. In order that the Euler characteristic be even, it is necessary that the the number of faces of every orientable imbedding be odd. Since H is upper-imbeddable, it follows that a maximum imbedding has only one face.

Suppose that some maximum imbedding of G_1 has the rotation

$$v_1.\ e_1, e_2, \ldots, e_r$$

and that some maximum imbedding of H has the rotation

$$u.\ f_1, f_2, \ldots, f_s.$$

We consider the imbedding of (G_2, v_2) formed by merging the vertices v_1 and u, so that the rotation at v_2 is

$$v_2.\ e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_s,$$

and all other rotations are as in the contributory imbeddings of G_1 and H. This merges the fb-walk of G_1 with the fb-walk of H, thereby creating an imbedding of G_2 with only one face, making it a maximum imbedding. Moreover, it follows from Euler-characteristic considerations that the genus of that maximum imbedding of G_2 is the sum $2\gamma_{\max}(H)$ of the genera of the two contributory imbeddings. Continuing inductively, we see that a maximum imbedding of each graph G_n in the H-linear sequence has one face, and is of genus $n\gamma_{\max}(H)$. □

Corollary 6.2. *Let (H, u, v) be an upper-imbeddable graph of even cycle rank, whose associated linear family $\{G_n\}$ is of type $(d_1, d_2, \ldots, d_k)$. Then we have $d_1 = \gamma_{\max}(H)$, and for $j = 2, 3, \ldots, k$, we have $d_j \leq j\gamma_{\max}(H)$.*

Proof. From Theorem 3.2, we have

$$\Gamma_{G_n}(z) \;=\; c_1(z)\Gamma_{G_{n-1}}(z) + c_2(z)\Gamma_{G_{n-2}}(z) + \cdots + c_k(z)\Gamma_{G_{n-k}}(z),$$

with coefficients $c_j(z)$ in the polynomial ring $\mathbb{Z}[z]$. It follows that

$$\deg(\Gamma_{G_n}) \;=\; \max\{\deg(c_j(z)) + \deg(\Gamma_{G_{n-j}}(z)) \mid j = 1, 2, \ldots, k\}.$$

By Theorem 6.1, we obtain

$$n\gamma_{\max}(H) \;=\; \max\{d_j + (n-j)\gamma_{\max}(H) \mid j = 1, 2, \ldots, k\},$$

and consequently,

$$n\gamma_{\max}(H) \;\geq\; d_j + (n-j)\gamma_{\max}(H), \text{ for } j = 1, 2, \ldots, k.$$

It follows that $d_j \leq j\gamma_{\max}(H)$, for $j = 1, 2, \ldots, k$. □

Remark. Theorem 6.1 and Corollary 6.2 could be adjusted so that the initial graph for the linear sequence is an arbitrary graph G_0 with suitable root-vertices.

An ***odd component*** of the edge-complement $G - T$ of a spanning tree T for a graph G is a component with an odd number of edges. The ***deficiency*** of a graph G is the minimum number $\xi(G)$ of odd components, taken over all spanning trees. A spanning tree that achieves this minimum is called a ***Xuong tree***. Xuong [Xu79] proved the following theorem:

Theorem 6.3 ([Xu79])**.** *Let G be any graph. Then*

$$\gamma_{\max}(G) \;=\; \frac{1}{2}\left(\beta(G) - \xi(G)\right). \tag{6.1}$$

We define ***property P***, whose presence or absence assigns any upper-imbeddable doubly rooted graph H to one of two categories:

- **P:** There is a Xuong tree T in H such that the odd component of $H - T$ is incident on both roots.

Theorem 6.4. *Let (H, u, v) be an upper-imbeddable graph in which the cycle rank $\beta(H)$ is odd. If H satisfies Property P, then for each graph G_n in the corresponding H-linear sequence $\{(G_n, v_n), n \geq 1\}$, we have*

$$\gamma_{\max}(G_n) \;=\; n\gamma_{\max}(H) + \left\lfloor \frac{n}{2} \right\rfloor. \tag{6.2}$$

Otherwise,

$$\gamma_{\max}(G_n) \;=\; n\gamma_{\max}(H). \tag{6.3}$$

Proof. Since the cycle rank of H is odd, it follows that $|V_H| - |E_H|$ is even. In order that the Euler characteristic be even, it is necessary that the number of faces of every orientable imbedding of H be even. Since H is upper-imbeddable, it follows that a maximum imbedding has two faces.

Let T be a Xuong tree for H. We consider the imbedding of (G_2, v_2) formed by amalgamating the root-vertex v_1 of G_1 with the root-vertex u of a copy of H. This merges the Xuong tree (T, v_1) of (G_1, v_1) with the Xuong tree (T, u) in (H, u), resulting in the amalgamated spanning tree $T *_{v_1=u} T$ (denoted $T * T$, for short) for the graph G_2.

Case 1. Suppose that H has property P. Then amalgamating G_1 and H also merges the only odd component of $G_1 - T$ with the only odd component of $H - T$, thereby yielding an even component of $G_2 - T * T$. It follows that $T * T$ is a Xuong tree for G_2 and that

$$\gamma_{\max}(G_2) = \frac{\beta(G_2)}{2} = 2\gamma_{\max}(H) + 1.$$

Next we consider the imbedding of (G_3, v_3) formed by amalgamating the root-vertex v_2 of G_2 with the root-vertex u of a copy of H. This merges the copy of the Xuong tree $T * T$ of G_2 with the copy of T in H, resulting in a spanning tree $T * T * T$ for the graph G_3. We observe that by the merger, the only odd component of $H - T$ becomes an odd component of $G_3 - T * T * T$. We see that $T * T * T$ is a Xuong tree for G_3 and that

$$\gamma_{\max}(G_3) = \frac{\beta(G_3)}{2} = 3\gamma_{\max}(H) + 1.$$

Continuing inductively, we see that a sequence of maximum imbeddings of the graphs G_n alternates between one-face and two-face imbeddings, and satisfies Equation (6.2).

Case 2. Suppose that H does not have property P. We observe that the graph $G_2 - T * T$ has two odd components, one in G_1, and the other in H. We assert that $T * T$ is a Xuong tree, which would imply that $\gamma_{\max}(G_2) = 2\gamma_{\max}(H)$. Suppose, to the contrary, that T' is a spanning tree for G_2 such that $G_2 - T'$ has no odd components. Then the spanning tree $T' \cap H$ for H has no odd components, which contradicts the assumption that T is a Xuong tree for H. Indeed, it follows by induction that the spanning tree $T^{(n)}$ — i.e., the result of amalgamating n copies of (T, u, v) — is a Xuong tree for $G - n$ and that its edge-complement in G_n has n odd components, one in each copy of H, along with Equation (6.3). □

Remark. Requiring that the odd component of $H - T$ be incident at both roots is necessary, as illustrated by the H-linear family in Figure 6.1.

We observe that H is upper-imbedable, that $\gamma_{\max}(H) = 1$, and that the odd complement of the only Xuong tree (thickened edges) is not incident at the roots u and v. We also observe that the graph G_2 is not upper-imbeddable.

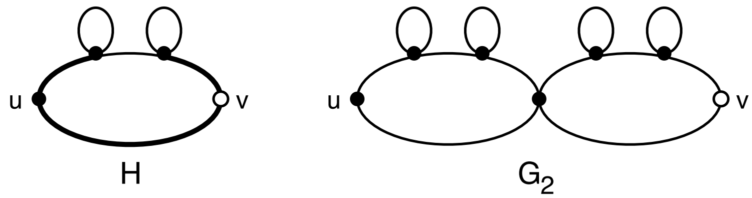

FIGURE 6.1. Chain of copies of a type-$(0,2)$-necklace.

Corollary 6.5. *Let (H,u,v) be an upper-imbeddable graph of odd cycle rank, whose associated linear family $\{G_n\}$ is of type $(d_1,d_2,\ldots,d_k)$. If H satisfies Property P, then for $j=1,2,\ldots,k$, we have $d_j \leq j\gamma_{\max}(H)+\left\lfloor\frac{j+1}{2}\right\rfloor$. Otherwise, for $j=1,2,\ldots,k$, we have $d_j \leq j\gamma_{\max}(H)$.*

Proof. From Theorem 3.2, we have

$$\Gamma_{G_n}(z) \;=\; c_1(z)\Gamma_{G_{n-1}}(z)+c_2(z)\Gamma_{G_{n-2}}(z)+\cdots+c_k(z)\Gamma_{G_k}(z),$$

with coefficients $c_j(z)$ in the polynomial ring $\mathbb{Z}[z]$. It follows that

$$\deg(\Gamma_{G_n}) \;=\; \max\{\deg(c_j(z))+\deg(\Gamma_{G_{n-j}}(z)) \mid j=1,2,\ldots,k\}.$$

If Property P holds, then by Equation (6.2), we obtain

$$n\gamma_{\max}(H)+\left\lfloor\frac{n}{2}\right\rfloor \;=\; \max\{d_j+(n-j)\gamma_{\max}(H) \mid j=1,2,\ldots,k\},$$

and consequently,

$$n\gamma_{\max}(H)+\left\lfloor\frac{n}{2}\right\rfloor \;\geq\; d_j+(n-j)\gamma_{\max}(H)+\left\lfloor\frac{n-j}{2}\right\rfloor \text{ for } j=1,2,\ldots,k.$$

It follows that $d_j \;\leq\; j\gamma_{\max}(H)+\left\lfloor\frac{j+1}{2}\right\rfloor$, for $j=1,2,\ldots,k$. Alternatively, if Property P does not hold, then by Equation (6.3), we have

$$n\gamma_{\max}(H) \;=\; \max\{d_j+(n-j)\gamma_{\max}(H) \mid j=1,2,\ldots,k\},$$

and consequently,

$$n\gamma_{\max}(H) \;\geq\; d_j+(n-j)\gamma_{\max}(H), \text{ for } j=1,2,\ldots,k.$$

It follows that $d_j \;\leq\; j\gamma_{\max}(H)$, for $j=1,2,\ldots,k$. □

7. RESEARCH QUESTIONS

Research Question 7.1. Recalling the remark at the end of Section 3, we ask the following: Given an H-linear sequence of graphs and a recurrence for their genus polynomials, is there a set of imbedding types for the graphs and a production matrix whose characteristic polynomial coefficients match the coefficients of the given recursion?

Research Question 7.2. Let G be a 3-connected graph and let e be an edge of G. Is it possible that there is no maximum-genus imbedding of G such that both sides of e lie on the same face? What if G is 2-connected, but not a cycle graph?

Research Question 7.3. We have explained why two imbedding types are sufficient when the rooted graph (H, U, V) satisfies either the premise of Theorem 4.1 or the premise of Theorem 5.1. We can generalize the premise of Theorem 5.1 by allowing the forward roots $u_1, u_2, \ldots, u_k$ to be 2-valent and to lie consecutively along a path in H. Are there any other restrictions on (H, U, V) such that two imbedding types would be sufficient?

Research Question 7.4. Suppose that $(d_1, d_2, \ldots, d_k)$ is the type of the linear recursion for the genus polynomials of a linear family of graphs. Is it necessary that $d_1 \leq d_2 \leq \cdots \leq d_k$? In general, the degrees of the coefficients of the characteristic polynomial in λ of a matrix over $\mathbb{Z}[z]$ need not be non-descending according to descending powers of λ. For instance, the 2×2 matrix

$$M(z) = \begin{bmatrix} 8 & 12 \\ 4z+12 & 6z+6 \end{bmatrix}$$

has $6z + 14$ (degree 1) as its trace and -96 (degree 0) as its determinant. We observe that both the column sums of the matrix $M(1)$ are 24, that is, exactly the same, as we have for any production matrix. Accordingly, a proof that the answer to the question is affirmative would have to follow from some other property of production matrices for linear graph sequences.

Remark. We observe that the choice of i-types can lead to a production matrix with a determinant of zero. That would imply that $k < \ell$ in Theorem 3.2, in which case we would have

$$c_{k+1}(z) = c_{k+2}(z) = \cdots = c_\ell(z) = 0.$$

In giving the type of the corresponding linear recurrence, we stop at d_k. Thus, this kind of example is not a negative answer to Research Question 7.4.

Research Question 7.5. We consider Research Question 7.4 for the special case $k = 2$. Is it always true in this circumstance that $d_1 \leq d_2$?

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