WEBSITE SUPPLEMENT TO
GENUS POLYNOMIALS OF LADDER-LIKE SEQUENCES OF GRAPHS

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4. General Production Matrix for Ladder-Like Graphs

The following theorem, our main theorem, presents a pair of 3×3 matrices and establishes a way to express the production matrix for any ladder-like sequence as a linear combination of those two matrices, such that the coefficients of the two matrices are the partial genus polynomials of the super-rung.

Theorem 4.1. *Let $(H, 0, 1)$ be any graph with two 1-valent root vertices. Let $p(z)$ and $q(z)$ be the partial genus polynomials for H of i-types (0)(1) and (01), respectively. Then the production matrix for the ladder-like sequence $L^H_1, L^H_2, L^H_3, \ldots$ is*

$$M_{L^H}(z) = p(z)\begin{bmatrix} 4z & 2z & 0 \\ 0 & 0 & 0 \\ 0 & 2z & 4z \end{bmatrix} + q(z)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 4z & 2z & 0 \end{bmatrix}. \tag{4.1}$$

Proof. In this Website Supplement, we give a proof with figures, similar to proofs appearing in many previous papers. The proof in the published version of the paper gives a completely symbolic calculation of the production matrix for a sequence of ladder-like graphs, using symbolic manipulation rules first described in [GKMT18].

There are three possible i-types for the ladder-like graph L^H_n and two possible i-types for the super-rung H. Every imbedding of L^H_{n+1} results from an imbedding of L^H_n and an imbedding of H. We will use three productions to describe the results of extending an imbedding of L^H_n by an imbedding of H of i-type (0)(1), and three more to describe the results of using an imbedding of H of i-type (01) to make the extension. Consequently, we need six productions

Yichao Chen is supported by the NNSFC under Grant No. 11471106.
Jonathan Gross is supported by Simons Foundation Grant No. 315001.
Thomas Tucker is supported by Simons Foundation Grant No. 317689.

to derive the production matrix $M_{L^H}(z)$ that maps the pgd-vector for L_n^H to the pgd-vector for L_{n+1}^H.

Each of the six figures of derivations of productions is actually a condensation of a derivation by string operations, as developed in [GKMT18]. We continue to represent the super-rung H by a blob with two pendant edges. At intermediate stages of the string-operations derivations, roots of L_n^H and roots of L_{n+1}^H may be present simultaneously.

Remark. To avoid confusion from any ambiguities, we use $\overline{0}$ and $\overline{1}$ to denote the roots of L_n^H, and we denote the root-vertices of H by 0 and 1.

i-type $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$. Figure 4.1 illustrates the derivation of the production for combining i-types $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$ and $(0)(1)$. Different colors are used to indicate different fb-walks. There are two stages to the derivation. We proceed from the initial configuration (at the left) to the intermediate configuration (in the middle) by joining root $\overline{0}$ of L_n^H by the edge e_1 to the root-vertex 0 of H. Then we proceed the from the intermediate configuration to the final configuration (at the right) by joining root $\overline{1}$ of L_n^H by the edge e_2 to the root-vertex 1 of H.

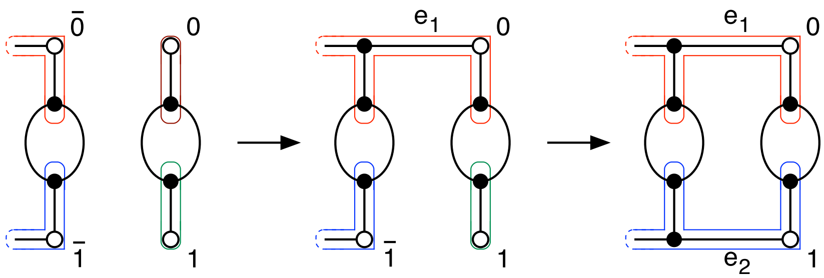

Figure 4.1. Combining i-types $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$ and $(0)(1)$ to get $(00)(11)$.

There are two locations at which the edge e_1 could be joined to the vertex $\overline{0}$, with the other location "inside the right angle". Since there are $p(z)$ imbeddings of H of i-type $(0)(1)$, then so far, from each imbedding of L_n^H of i-type $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$, we have $2p(z)$ imbeddings of i-type $(00)(\overline{1}\,\overline{1})(1)$ for the intermediate configuration (in the middle drawing of Figure 4.1). In that drawing, we have suppressed the label at root $\overline{0}$ and we have replaced the hollow dot, which indicates a root, by a solid dot, which indicates a non-root.

Similarly, when we join the the root $\overline{1}$ to the root 1 by the edge e_2, there are two locations at which e_2 could be joined to root 1. For either location, the resulting imbedding has the same type. We observe that adding edge e_2 to the middle configuration reduces the number of faces by one. Thus, the genus of the surface increases by one. The second edge-addition yields two imbeddings of type $(00)(11)$. In summary, we have derived the production

$$(\overline{0}\,\overline{0})(\overline{1}\,\overline{1}) \to 4zp(z)(00)(11). \tag{4.2}$$

Figure 4.2 illustrates the derivation of the production for combining i-types $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$ and (01).

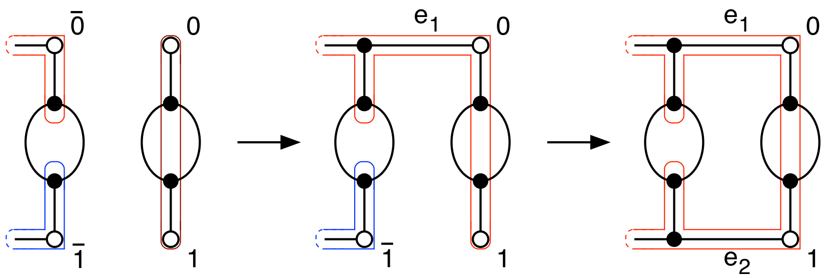

FIGURE 4.2. Combining i-types $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$ and (01) to get (0011).

Joining an imbedding of i-type $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$ to $q(z)$ imbeddings of i-type (01) by edge e_1 gives us $2q(z)$ imbeddings of i-type $(001)(\overline{1}\,\overline{1})$. Then adding edge e_2 gives two imbeddings of i-type (0011), for each imbedding of i-type $(001)(\overline{1}\,\overline{1})$. This results in the following production:

$$(\overline{0}\,\overline{0})(\overline{1}\,\overline{1}) \to 4zq(z)(0011). \tag{4.3}$$

We can combine the productions (4.2) and (4.3) to form the combined production that describes the effect of adding the next copy of the super-rung H to an imbedding of i-type $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$:

$$(\overline{0}\,\overline{0})(\overline{1}\,\overline{1}) \to 4zp(z)(00)(11) + 4zq(z)(0011). \tag{4.4}$$

i-type $(\overline{0}\,\overline{1})(\overline{0}\,\overline{1})$. We now consider joining an imbedding of L_n^H of i-type $(\overline{0}\,\overline{1})(\overline{0}\,\overline{1})$ to an imbedding of H of i-type (0)(1). When we join vertices 0 and $\overline{0}$ by edge e_1, we get $2p(z)(00\overline{1})(\overline{1})(1)$.

Adding edge e_2 is slightly trickier, because the two different locations at which to attach edge e_2 to the root-vertex $\overline{1}$ yield imbeddings of different types, as indicated by Figure 4.3. An imbedding of i-type $(00\overline{1})(\overline{1})(1)$ transforms into one imbedding of i-type (00)(11) and another of i-type (0011). This is the resulting production:

$$(\overline{0}\,\overline{1})(\overline{0}\,\overline{1}) \to 2zp(z)(00)(11) + 2zp(z)(0011). \tag{4.5}$$

The consequent of the production for combining i-type $(\overline{0}\,\overline{1})(\overline{0}\,\overline{1})$ for L_n^H with i-type (01) for H also splits into two cases, as shown in Figure 4.4. Here the production is

$$(\overline{0}\,\overline{1})(\overline{0}\,\overline{1}) \to 2q(z)(01)(01) + 2zq(z)(0011). \tag{4.6}$$

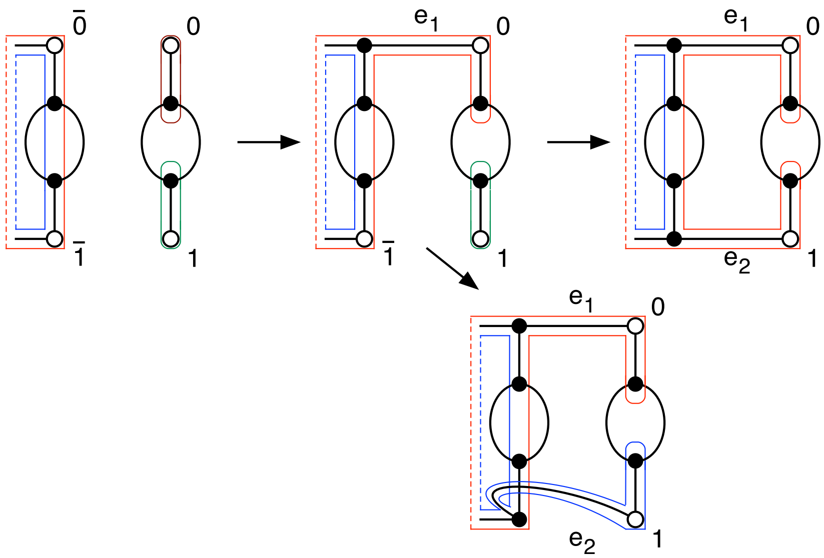

FIGURE 4.3. Combining $(\bar{0}\,\bar{1})(\bar{0}\,\bar{1})$ and (0)(1) to get (0011) and (00)(11).

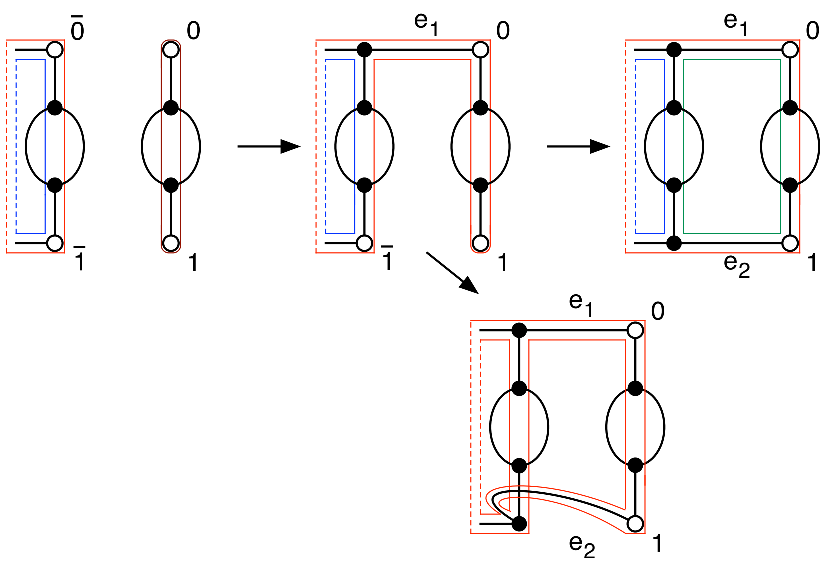

FIGURE 4.4. Combining $(\bar{0}\,\bar{1})(\bar{0}\,\bar{1})$ and (01) to get (01)(01) and (0011).

When we combine (4.5) and (4.6), the resulting complete production for adding the super-rung H to an i-type $(\bar{0}\,\bar{1})(\bar{0}\,\bar{1})$ imbedding is

$$(\bar{0}\,\bar{0})(\bar{1}\,\bar{1}) \to 2zp(z)(00)(11) + 2q(z)(01)(01) + 2z(p(z)+q(z))(0011). \tag{4.7}$$

i-type $(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})$. Combining i-type $(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})$ for L_n^H with i-type (0)(1) for H is illustrated by Figure 4.5. We have the following production:

$$(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1}) \to 4zp(z)(0011). \tag{4.8}$$

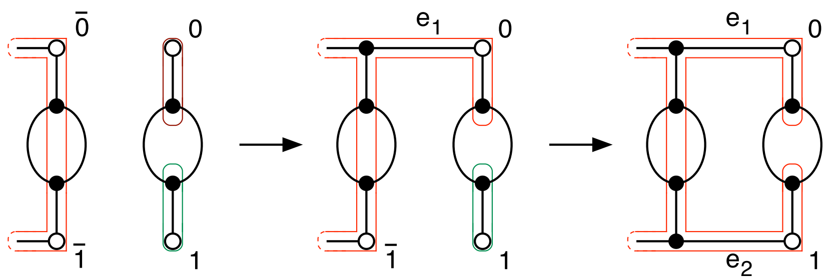

FIGURE 4.5. Combining i-types $(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})$ and (0)(1) to get (0011).

The last case to consider is combining i-type $(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})$ for L_n^H with i-type (01) for H. As shown by Figure 4.6, the consequent has only one term.

$$(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1}) \to 4q(z)(01)(01). \tag{4.9}$$

The complete production for i-type $(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})$ is

$$(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1}) \to 4q(z)(01)(01) + 4zp(z)(0011). \tag{4.10}$$

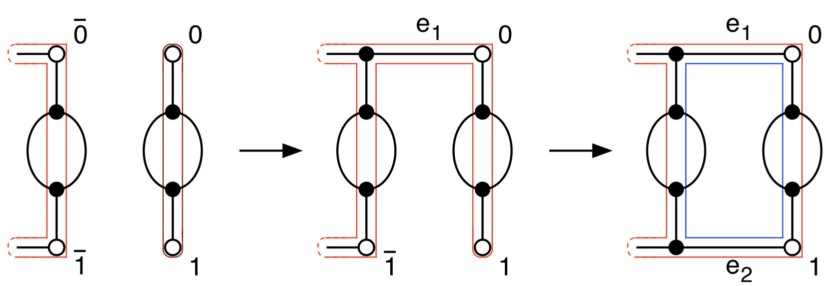

FIGURE 4.6. Combining i-types $(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})$ and (01) to get (01)(01).

We can represent the three productions (4.4), (4.7), and (4.10) as the columns of a single production matrix $M_{L^H}(z)$ that maps the pgd-vector for L_n^H to the pgd-vector for L_{n+1}^H.

$$M_{L^H}(z) = \begin{bmatrix} 4zp(z) & 2zp(z) & 0 \\ 0 & 2q(z) & 4q(z) \\ 4zq(z) & 2zp(z) + 2zq(z) & 4zp(z) \end{bmatrix}$$

Splitting the matrix $M_{L^H}(z)$ into the following sum completes the first proof.

$$M_{L^H}(z) \;=\; p(z)\begin{bmatrix} 4z & 2z & 0 \\ 0 & 0 & 0 \\ 0 & 2z & 4z \end{bmatrix} \;+\; q(z)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 4z & 2z & 0 \end{bmatrix} \qquad \square$$

References

[GKMT18] J.L. Gross, I.F. Khan, T. Mansour, and T.W. Tucker, Calculating genus polynomials via string operations and matrices, *Ars Math. Contemporanea* **15** (2018) 267–295. Presented at the Eighth Slovenian Graph Theory Conference, 2015.

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