Question 1

Substituting the optimal values for e and q into $Q(C, e, q)$ gives

$$
\begin{aligned}
Q(C) &= \sum_{u,v} f(u,v) \left[\log \frac{f_2(v)}{g_2(C(v))} + \log \frac{g(C(u), C(v))}{g_1(C(u))} \right] \\
&= \sum_{u,v} f(u,v) \left[\log f_2(v) + \log \frac{g(C(u), C(v))}{g_1(C(u)) g_2(C(v))} \right] \\
&= G + \sum_{u,v} f(u,v) \log \frac{g(C(u), C(v))}{g_1(C(u)) g_2(C(v))} \\
&= G + \sum_{c} \sum_{c'} \sum_{u:C(u)=c, v:C(v)=c'} f(u,v) \log \frac{g(C(u), C(v))}{g_1(C(u)) g_2(C(v))} \\
&= G + \sum_{c} \sum_{c'} \sum_{u:C(u)=c, v:C(v)=c'} f(u,v) \log \frac{g(c, c')}{g_1(c) g_2(c')} \\
&= G + \sum_{c} \sum_{c'} g(c, c') \log \frac{g(c, c')}{g_1(c) g_2(c')}
\end{aligned}
$$

Question 2

$$
\begin{aligned}
& L(\Theta', \Theta) \\
= & \sum_{u,v} \left[p(u,v) \log \frac{\exp\{\theta'_u \cdot \theta_v\}}{1 + \exp\{\theta'_u \cdot \theta_v\}} + K p_1(u) p_2(v) \log \frac{1}{1 + \exp\{\theta'_u \cdot \theta_v\}} \right] \\
= & \sum_{u,v} \alpha(u,v) \left[\frac{p(u,v)}{\alpha(u,v)} \log \frac{\exp\{\theta'_u \cdot \theta_v\}}{1 + \exp\{\theta'_u \cdot \theta_v\}} + \frac{K p_1(u) p_2(v)}{\alpha(u,v)} \log \frac{1}{1 + \exp\{\theta'_u \cdot \theta_v\}} \right]
\end{aligned}
$$

where $\alpha(u,v) = p(u,v) + K p_1(u) p_2(v)$

Question 2

$$L(\Theta', \Theta) = \sum_{u,v} \alpha(u,v) \left[\frac{p(u,v)}{\alpha(u,v)} \log \frac{\exp\{\theta'_u \cdot \theta_v\}}{1 + \exp\{\theta'_u \cdot \theta_v\}} + \frac{K p_1(u) p_2(v)}{\alpha(u,v)} \log \frac{1}{1 + \exp\{\theta'_u \cdot \theta_v\}} \right]$$

where $\alpha(u,v) = p(u,v) + K p_1(u) p_2(v)$

Next note that if Θ', Θ is such that for all u, v,

$$\theta'_u \cdot \theta_v = \log \frac{p(u,v)}{p_1(u) p_2(v)} - \log K \Rightarrow \exp\{\theta'_u \cdot \theta_v\} = \frac{p(u,v)}{K p_1(u) p_2(v)}$$

then it follows that

$$\frac{\exp\{\theta'_u \cdot \theta_v\}}{1 + \exp\{\theta'_u \cdot \theta_v\}} = \frac{p(u,v)}{\alpha(u,v)} \quad \text{and} \quad \frac{1}{1 + \exp\{\theta'_u \cdot \theta_v\}} = \frac{K p_1(u) p_2(v)}{\alpha(u,v)}$$

Hence this value for Θ', Θ maximizes the $[\ldots]$ term above for each value of (u, v). It follows that any maximizer of $L(\Theta', \Theta)$ must be such that for all u, v, $\theta'_u \cdot \theta_v = \log \frac{p(u,v)}{p_1(u) p_2(v)} - \log K$. Otherwise there is some $[\ldots]$ term that is not maximized.