GENUS DISTRIBUTIONS OF STAR-LADDERS

YICHAO CHEN, JONATHAN L. GROSS, AND TOUFIK MANSOUR

Abstract. Star-ladder graphs were introduced by Gross in his development of a quadratic-time algorithm for the genus distribution of a cubic outerplanar graph. This paper derives a formula for the genus distribution of star-ladder graphs, using Mohar's overlap matrix and Chebyshev polynomials.

Newly developed methods have led to a number of recent papers that derive genus distributions and total embedding distributions for various families of graphs. Our focus here is on a family of graphs called *star-ladders*.

1. Introduction

Genus distributions problems have frequently been investigated in the past quarter century, since the topic was inaugurated by Gross and Furst [6]. The contributions include [1, 5, 8, 9, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 26] and [27]. Gross [11] presents a quadratic-time algorithm for computing the genus distribution of any cubic outerplanar graph. He analyzes the structure of any cubic outerplanar graph and finds that such a graph can be obtained by a series of iterated edge-amalgamations of a new classes of graphs called star-ladders, so as to form a tree of star-ladders. Thus, beyond the direct interest in a closed formula for the genus distribution of star-ladders, such a formula is possibly a step toward a closed formula for the genus distribution of the cubic outerplanar graphs. Our closed formula in this paper for the genus distribution of star-ladders is derived with the aid of Mohar's overlap matrices [17].

1.1. Star-ladders.

An ***n-rung closed-end ladder*** L_n can be obtained by taking the graphical cartesian product of an *n-vertex* path with the complete graph K_2, and then doubling both its end edges. The new rungs obtained thereby are called end-rungs. Figure 1 presents a 4-rung closed-end ladder. In [5], Furst, Gross, and Statman obtained a closed formula for the genus distribution of closed-end ladders.

2000 *Mathematics Subject Classification.* Primary: 05C10; Secondary: 30B70, 42C05.

Key words and phrases. graph embedding; genus distribution; star-ladders; overlap matrix; Chebyshev polynomials.

The work of the first author was partially supported by NNSFC under Grant No. 10901048.

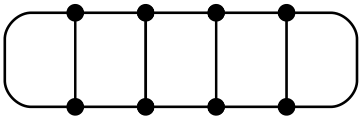

FIGURE 1. The 4-rung closed-end ladder L_4.

For an k-tuple of non-negative integers $U = (n_1, n_2, \dots, n_k)$ the ***star-ladder*** with signature U is the graph $SL_{n_1,n_2,\dots,n_k}$ obtained from the cycle graph C_{2k}, with consecutive edges labeled $e_1, e_2, \dots, e_{2k}$ as follows:

(1) For each $i \le k$ such that $n_i = 0$, double the edge e_{2i}.
(2) For each of the ladders $L_{n_1}, L_{n_2}, \dots, L_{n_k}$ such that $n_i > 0$,
  - subdivide one end rung of L_{n_i} into three parts, and take the middle third as the root-edge;
  - amalgamate L_{n_i} across its newly created root edge to the edge e_{2i}.

The star-ladder $SL_{2,1,0}$ is shown in Figure 2.

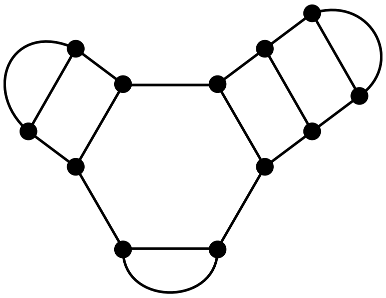

FIGURE 2. The star-ladder $SL_{2,1,0}$.

1.2. **Genus polynomial.** It is assumed that the reader is somewhat familiar with the basics of topological graph theory, as found in Gross and Tucker [7]. All graphs considered in this paper are connected. A ***graph*** $G = (V(G), E(G))$ is permitted to have both loops and multiple edges. A ***surface*** is a compact 2-manifold without boundary. In topology, surfaces are classified into the *orientable surfaces* S_g, with g handles ($g \ge 0$), and the *nonorientable surfaces* N_k, with k crosscaps ($k > 0$). A ***graph embedding*** into a surface means a *cellular embedding.* For any spanning tree of G, the number of co-tree edges is called the ***Betti number*** of G, and is denoted by $\beta(G)$.

A ***rotation at a vertex*** v of a graph G is a cyclic order of all edge-ends (or equivalently, half-edges) incident with v. A ***pure rotation system*** ρ of a graph G is the collection of rotations at all vertices of G. An embedding of G into an oriented surface S induces a pure rotation system as follows: the rotation at v is the cyclic permutation corresponding to the order in which the edge-ends are traversed in an orientation-preserving tour around v. Conversely, by the

Heffter-Edmonds principle, every rotation system induces a unique embedding (up to homeomorphism) of G into some orientable surface S. The bijection of this correspondence implies that the total number of orientable embeddings is

$$\prod_{v \in V(G)} (d_v - 1)!,$$

where d_v is the degree of vertex v.

A ***general rotation system*** is a pair (ρ, λ), where ρ is a pure rotation system and λ is a mapping $E(G) \to \{0, 1\}$. The edge e is said to be *twisted* (respectively, *untwisted*) if $\lambda(e) = 1$ (respectively, $\lambda(e) = 0$). It is well-known that every oriented embedding of a graph G can be described by a general rotation system (ρ, λ) with $\lambda(e) = 0$ for all $e \in E(G)$. By allowing λ to take non-zero values, we can describe the nonorientable embeddings of G. For any spaning tree T, a ***T-rotation system*** (ρ, λ) of G is a general rotation system (ρ, λ) such that $\lambda(e) = 0$, for all $e \in E(T)$.

By the ***genus polynomial of a graph*** G, we mean the polynomial

$$\Gamma_G(z) \;=\; \sum_{i=0}^{\infty} g_i(G) z^i,$$

where $g_i(G)$ means the number of embeddings of G into the orientable surface S_i, for $i \geq 0$.

1.3. **Overlap matrices.** Mohar [17] introduced an invariant that has subsequently been used numerous times (e.g., [2, 3, 4]) in the calculation of distributions of graph embeddings, including non-orientable embeddings. We use Mohar's invariant here in our derivation of a formula for the genus distribution of star-ladders.

Let T be a spanning tree of a graph G and let (ρ, λ) be a T-rotation system. Let $e_1, e_2, \ldots, e_{\beta(G)}$ be the cotree edges of T, where $\beta(G)$ is the cycle rank of G. The ***overlap matrix*** of (ρ, λ) is the $\beta(G) \times \beta(G)$ matrix $M = [m_{ij}]$ over $\mathbb{Z}_2$ such that

$$m_{ij} \;=\; \begin{cases} 1, & \text{if } i = j \text{ and } e_i \text{ is twisted;} \\ 1, & \text{if } i \neq j \text{ and the restriction of the underlying pure} \\ & \text{rotation system to the subgraph } T + e_i + e_j \text{ is nonplanar;} \\ 0, & \text{otherwise.} \end{cases}$$

When the restriction of the underlying pure rotation system to the subgraph $T + e_i + e_j$ is nonplanar, we say that edges e_i and e_j ***overlap***. The importance of overlap matrices is indicated by this theorem of Mohar [17]:

Theorem 1.1. *Let (ρ, λ) be a general rotation system for a graph. Then the rank of any overlap matrix M for the corresponding embedding equals twice the genus of the embedding surface, if that surface is orientable, and it equals the crosscap number otherwise. The rank is independent of the choice of a spanning tree.*

For drawing a planar representation of a rotation system on a cubic graph, we adopt the graphic convention introduced by Gustin [13], and used extensively by Ringel and Youngs (see [21]) in their solution to the Heawood map-coloring problem. There are two possible cyclic orderings of each trivalent vertex. Under this convention, we color a vertex *black*, if the rotation of the edge-ends incident on it is *clockwise*, and we color it *white* if the rotation is *counterclockwise*. We call any drawing of a graph that uses this convention to indicate a rotation system a ***Gustin representation*** of that rotation system.

The approach here is similar approach to that used for ladders in [5]. In a Gustin representation of a rotation system for a graph, an edge is called ***matched*** if it has the same color at both endpoints; otherwise, it is called ***unmatched***. In Figure 3, we have indicated our choice of a spanning tree for a generic ladder L_{n-1} by thicker lines and a partial choice of rotations at the vertices.

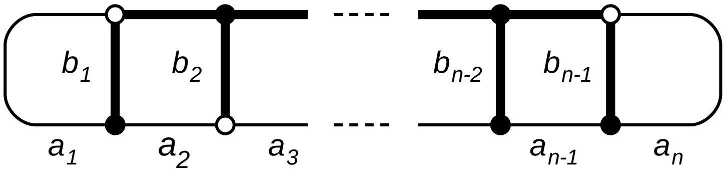

FIGURE 3. A spanning tree and some rotations for the ladder L_{n-1}.

The following proposition facilitates the calculation of an overlap matrix for a ladder graph. The proof is simply to apply the Heffter-Edmonds face-tracing algorithm.

Proposition 1.2. *In the ladder L_{n-1}, we choose all of the edges on one side of the ladder plus all of the rungs, except for the two created by doubling, as the edges of a spanning tree. We label the cotree edges $a_1, \ldots, a_n$, from one end of the ladder to the other, and we label the tree rungs $b_1, \ldots, b_n$, from one end of the ladder to the other (as shown in Figure 3). Then two cotree edges a_i and a_{i+1}, with $1 \leq i \leq n-1$, overlap if and only if the rung edge b_i is unmatched.*

By Proposition 1.2, the overlap matrix M_n of L_{n-1} can be written as

$$M_n \;=\; M_n^{X,Y} \;=\; \begin{pmatrix} x_1 & y_1 & & & & \\ y_1 & x_2 & y_2 & & \mathbf{0} & \\ & y_2 & x_3 & y_3 & & \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ & & & & y_{n-1} & x_n \end{pmatrix},$$

where $X = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n$ and $Y = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{Z}_2^{n-1}$. Our notation $M_n^{X,Y}$ indicates not only that this is a tridiagonal $n \times n$ matrix, but also that the diagonal arrays just below and just above the main diagonal are identical. We say that such a matrix is ***symmetrically tridiagonal***.

Corollary 1.3. *Each symmetrically tridiagonal matrix $M_n^{X,Y}$ corresponds to exactly 2^{n-1} different T-rotation systems for the ladder L_{n-1}, where T is the spanning tree of L_{n-1} given in Figure 3.*

Proof. According to Proposition 1.2, changing the rotations at both endpoints of any or all of the rungs b_j does not change any of the coefficients in the overlap matrix. Moreover, any other change of rotations in ρ does change the overlap matrix. □

1.4. **The rank-distribution polynomial.** We now consider the set

$$\mathcal{A}_n \;=\; \{M_n^{X,Y} \mid X \in \mathbb{Z}_2^n \text{ and } Y \in \mathbb{Z}_2^{n-1}\},$$

of all symmetrically tridiagonal $n \times n$ matrices over $\mathbb{Z}_2$. We define the ***rank-distribution polynomial*** of the set $\mathcal{A}_n$ to be the polynomial

$$P_n(z) \;=\; \sum_{j=0}^{n} C_n(j) z^j,$$

where $C_n(j)$ is the number of different assignments of the variables x_i and y_k, with $1 \le i \le n$ and $1 \le k \le n-1$, for which the matrix $M_n^{X,Y}$ in $\mathcal{A}_n$ has rank j. Similarly, we consider the set

$$\mathcal{O}_n \;=\; \{M_n^{0,Y} \mid Y \in \mathbb{Z}_2^{n-1}\},$$

and we define the ***rank-distribution polynomial*** of $\mathcal{O}_n$ to be the polynomial

$$O_n(z) \;=\; \sum_{j=0}^{n} B_n(j) z^j, \tag{1}$$

where $B_n(j)$ is the number of different assignment of the variables $y_1, \ldots, y_{n-1}$ for which the matrix $M_n^Y = M_n^{0,Y}$ in $\mathcal{O}_n$ has rank j.

We recall that the ***Chebyshev polynomials of the second kind*** are defined by

$$U_n(t) \;=\; 2tU_{n-1}(t) - U_{n-2}(t), \quad U_0(t) = 1, \quad U_1(t) = 2t. \tag{2}$$

Lemma 1.4. *The rank-distribution polynomial* $O_n(z)$ *for symmetrically tridiagonal* $n \times n$ *matrices satisfies the recurrence relation*

$$O_n(z) \;=\; O_{n-1}(z) \;+\; 2z^2 O_{n-2}(z) \tag{3}$$

with the initial conditions

$$O_0(z) = O_1(z) = 1 \text{ and } O_2(z) = z^2 + 1. \tag{4}$$

Moreover,

$$O_n(z) \;=\; (iz\sqrt{2})^n \left[U_n\left(\frac{1}{2iz\sqrt{2}}\right) + \frac{1}{2} U_{n-2}\left(\frac{1}{2iz\sqrt{2}}\right)\right] \tag{5}$$

where $i^2 = -1$*, and where* U_m *is the* m^{th} *Chebyshev polynomial of the second kind.*

Proof. It is directly ascertainable that the sequence of functions $B_n(j)$ satisfies the recurrence system

$$\begin{aligned} B_0(j) &= 0 \quad \text{for } j \neq 0 \\ B_n(0) &= 1 \quad \text{for all } n = 0, 1, \ldots \\ B_2(2) &= 1 \\ B_n(j) &= B_{n-1}(j) \;+\; 2B_{n-2}(j-2). \end{aligned} \tag{6}$$

It follows, in turn, from its definition (1) that the polynomial $O_n(z)$ satisfies the recursion (3) and the initial conditions (4). Applying induction on n to the recursion (3), while using the Chebyshev recursion (2), we obtain equation (5):

$$O_n(z) \;=\; (iz\sqrt{2})^n \left[U_n\left(\frac{1}{2iz\sqrt{2}}\right) + \frac{1}{2}\, U_{n-2}\left(\frac{1}{2iz\sqrt{2}}\right)\right],$$

which completes the proof. □

Theorem 1.5. *(Furst et al.[5]) The number of embeddings of the closed-end ladder* L_{n-1} *into the orientable surface* S_i *is*

$$g_i(L_{n-1}) \;=\; \begin{cases} 2^{n-2+i}\binom{n-i}{i}\frac{2n-3i}{n-i}, & \text{when } i \leq \lfloor \frac{n}{2} \rfloor \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let the genus polynomial of the ladder L_{n-1} be

$$\Gamma_{L_{n-1}}(z) \;=\; \sum_{i \geq 0} g_i(L_{n-1}) z^i.$$

By Formula (5) of Lemma 1.4 and Corollary 1.3, we have

$$\begin{aligned}\Gamma_{L_{n-1}}(z) &= 2^{n-1}O_n(z)\\ &= 2^{n-1}\left\{\sum_{j\geq 0}\binom{n-j}{j}2^j\ z^{2j} - \sum_{j\geq 0}\binom{n-2-j}{j}2^j\ z^{2j+2}\right\}. \qquad (7)\end{aligned}$$

Note that $g_j(L_{n-1})$ is equal to the coefficient of z^{2j}. By (7), we have

$$g_j(L_{n-1}) = 2^{n-1}\left\{\binom{n-j}{j}2^j - \binom{n-j-1}{j-1}2^{j-1}\right\}.$$

By Newton's identity $\binom{n-m}{m} = \frac{n-m}{m}\binom{n-m-1}{m-1}$, the theorem follows. □

2. Rank-Distribution Polynomial of Star-Ladders

We fix a spanning tree T of $SL_{n_1,n_2,\ldots,n_k}$, shown by thicker lines in Figure 4, with cotree edges $e, e_{1,0}, e_{1,1}, \ldots, e_{1,n_1}$, $e_{2,0}, e_{2,1}, \ldots, e_{2,n_2}$, $\ldots$, $e_{k,0}, e_{k,1}, \ldots, e_{k,n_k}$, also as shown.

Property 2.1. *The cotree edge e overlaps the cotree edge $e_{i,0}$ if and only if the edge $b_{i,0}$ is unmatched, for $i = 1, 2, \cdots, k$.*

Property 2.2. *The cotree edges e_{i,i_j} and e_{i,i_j+1} overlap if and only if the edge b_{i,i_j+1} is unmatched, for $i = 1, 2, \cdots, k, i_j = 0, 1, \cdots, n_i - 1$.*

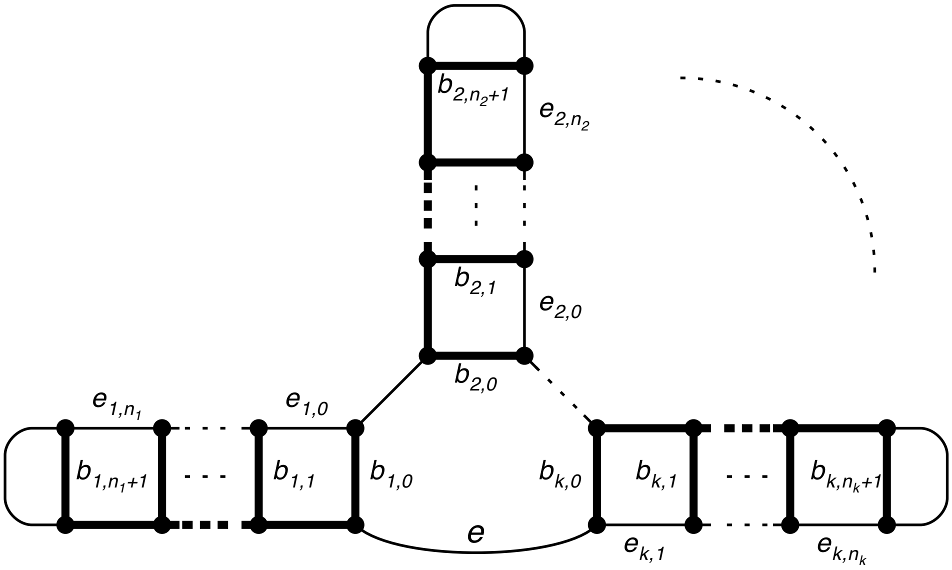

FIGURE 4. A spanning tree for the star-ladder SL_{n_1,n_2,n_k}

Let $W = (w_1, w_2, \ldots, w_k) \in \mathbb{Z}_2^n$ and $Y_i = (y_{i_1}, y_{i_2}, \ldots, y_{i_{k_i}}) \in \mathbb{Z}_2^{k_i}$, for $i = 1, 2, \cdots, k$. Then the overlap matrix of a star-ladder $SL_{n_1,n_2,\ldots,n_k}$ can be written in the following form:

$$M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k} =$$

$$\left(\begin{array}{ccccccccccccccccc}
0 & w_1 & 0 & 0 & \cdots & 0 & w_2 & 0 & 0 & \cdots & 0 & \cdots & w_k & 0 & 0 & \cdots & 0 \\
w_1 & 0 & y_{1,1} & & & & & & & & & & & & & & \\
0 & y_{1,1} & 0 & y_{1,2} & & & & & & & & & & & & & \\
0 & & y_{1,2} & \ddots & \ddots & & & & & & & & & & & & \\
\vdots & & & \ddots & & y_{1,n_1} & & & & & & & & & & & \\
0 & & & & y_{1,n_1} & 0 & 0 & & & & & & & & & & \\
w_2 & & & & & 0 & 0 & y_{2,1} & & & & & & & & & \\
0 & & & & & & y_{2,1} & 0 & y_{2,2} & & & & & & & & \\
0 & & & & & & & y_{2,2} & \ddots & \ddots & & & & & & & \\
\vdots & & & & & & & & \ddots & & y_{2,n_2} & & & & & & \\
0 & & & & & & & & & y_{2,n_2} & 0 & & & & & & \\
\vdots & & & & & & & & & & \ddots & \ddots & \ddots & & & & \\
w_k & & & & & & & & & & & & 0 & y_{k1} & & & \\
0 & & & & & & & & & & & & y_{k,1} & 0 & y_{k,2} & & \\
0 & & & & & & & & & & & & & y_{k,2} & \ddots & \ddots & \\
\vdots & & & & & & & & & & & & & & \ddots & & y_{k,n_k} \\
0 & & & & & & & & & & & & & & & y_{k,n_k} & 0
\end{array}\right),$$

where $w_i = 1$, for $i = 1, 2, \ldots, k$, if and only if $b_{i,0}$ is unmatched, and where $y_{i,i_k} = 1$, for $i = 1, 2, \ldots, k$ and $i_k = 1, 2, \ldots, n_i$, if only if b_{i,i_k} is unmatched.

Proposition 2.3. *For a fixed overlap matrix of the form* $M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k}$*, corresponding to the spanning tree* T *in a star-ladder graph* $SL_{n_1,n_2,\ldots,n_k}$*, there are exactly* $2^{\sum_{i=1}^{k}(n_i+1)}$ *different* T*-rotation systems corresponding to that matrix.*

Proof. This proof is like that of Property 1.3. ☐

We let $\mathcal{S}_{n_1,n_2,\ldots,n_k}$ denote the set of all matrices over $\mathbb{Z}_2$ that are of the type $M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k}$. We let $D(j)$ denote the number of different assignments of the variables w_j, y_{i,i_k} for which the matrix $M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k}$ in $\mathcal{S}_{n_1,n_2,\ldots,n_k}$ has rank j, where $j = 1, 2, \cdots, n$; $i = 1, 2, \cdots, k$; and $i_k = 1, 2, \ldots, n_i$.

Additionally, We define the ***rank-distribution polynomial***

$$\mathbb{S}_{n_1,n_2,\ldots,n_k}(z) = \sum_{j=0}^{n_1+n_2+\cdots+n_k+k+1} D(j) z^j. \tag{8}$$

Theorem 2.4. *The rank-distribution polynomial* $\mathcal{S}_{n_1,n_2,\ldots,n_k}(z)$ *of the overlap matrices of a star-ladder graph* $SL_{n_1,n_2,\ldots,n_k}$*satisfies the recurrence relation*

$$\mathcal{S}_{n_1,n_2,\ldots,n_k}(z) \;=\; \mathcal{S}_{n_1,n_2,\ldots,n_k-1}(z) + 2z^2\mathcal{S}_{n_1,n_2,\ldots,n_k-2}(z)$$

with the initial conditions

$$\mathcal{S}_{n_1,n_2,\ldots,n_{k-1},0}(z) \;=\; \mathcal{S}_{n_1,n_2,\ldots,n_{k-1}}(z) + 2^{k-1}z^2 O_{n_1+1}(z)O_{n_2+1}(z)\cdots O_{n_{k-1}+1}(z)$$

and

$$\mathcal{S}_{n_1,n_2,\ldots,n_{k-1},1}(z) \;=\; \mathcal{S}_{n_1,n_2,\ldots,n_{k-1},0}(z) + 2z^2\mathcal{S}_{n_1,n_2,\ldots,n_{k-1}}(z),$$

where $O_m(z)$ *is the rank distribution polynomial of the overlap matrices of the ladder graph* L_{m-1}*, as defined in Equation* (1)*.*

Proof. There are two cases.

Case 1. For $y_{n_k} = 0$. It is clear that

$$rank(M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k}) = rank(M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k-1})$$

so it contributes a term $\mathcal{S}_{n_1,n_2,\ldots,n_k-1}(z)$.

Case 2. For $y_{n_k} = 1$. If $y_{n_k-1} = 0$, then

$$rank(M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k}) = 2 + rank(M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k-2}).$$

Otherwise $y_{n_k-1} = 1$, under which circumstance we add the last row and last column, respectively, to row $n_1+n_2+\ldots+n_k+k$ and to column $n_1+n_2+\ldots+n_k+k$. We see thereby that $rank(M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k})$ is equal to 2 plus the rank of the upper-left matrix, which has the form of $M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k-2}$, that is,

$$rank(M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k}) = 2 + rank(M^{W,Y_1,Y_2,\cdots,Y_k}_{n_1,n_2,\ldots,n_k-2}).$$

In total, it contributes a term $2z^2\mathcal{S}_{n_1,n_2,\ldots,n_k-2}(z)$.

Hence, the polynomials $\mathcal{S}_{n_1,n_2,\ldots,n_k}(z)$ satisfy the recurrence relation

$$\mathcal{S}_{n_1,n_2,\ldots,n_k}(z) \;=\; \mathcal{S}_{n_1,n_2,\ldots,n_k-1}(z) + 2z^2\mathcal{S}_{n_1,n_2,\ldots,n_k-2}(z),$$

for all $n_k \geq 2$ and $k \geq 3$. □

Note that for $k = 2$, the definition (8) implies that

$$\mathcal{S}_{n_1,n_2}(z) = \sum_{j=0}^{n_1+n_2+3} D(j)z^j = O_{n_1+n_2+2}(z),$$

where

$$O_n(z) \;=\; \sum_{j\geq 0}\binom{n-j}{j}2^j\,z^{2j} - \sum_{j\geq 0}\binom{n-2-j}{j}2^j\,z^{2j+2}.$$

Moreover, for $k=3$, Theorem 2.4 implies these three equations:

$$\begin{aligned}
\mathcal{S}_{n_1,n_2,n_3}(z) &= \mathcal{S}_{n_1,n_2,n_3-1}(z)+2z^2\mathcal{S}_{n_1,n_2,n_3-2}(z)\\
\mathcal{S}_{n_1,n_2,0}(z) &= O_{n_1+n_2+2}(z)+4z^2O_{n_1+1}(z)O_{n_2+1}(z)\\
\mathcal{S}_{n_1,n_2,1}(z) &= \mathcal{S}_{n_1,n_2,0}(z)+2z^2O_{n_1+n_2+2}(z).
\end{aligned}$$

To solve the recursion of Theorem 2.4, we define

$$\mathcal{S}(t_1,t_2,\ldots,t_k,z) = \sum_{n_1,n_2,\ldots,n_k\geq 0}\mathcal{S}_{n_1,n_2,\ldots,n_k}(z)\,t_1^{n_1}t_2^{n_2}\cdots t_k^{n_k} \tag{9}$$

and

$$O(t,z) = \sum_{n\geq 1}O_n(z)t^n. \tag{10}$$

Rewriting the recurrence relation in the statement of Lemma 1.4 in terms of a generating function, we obtain

$$O(t,z) = \frac{(1+z^2t)t}{1-t-2z^2t^2}. \tag{11}$$

Rewriting the recurrence in the statement of Theorem 2.4 as a generating function, we obtain

$$\begin{aligned}
\mathcal{S}(t_1,t_2,\ldots,t_k,z) &= \mathcal{S}(t_1,t_2,\ldots,t_{k-1},z)\\
&\quad+2^{k-1}z^2\prod_{j=1}^{k-1}t_j^{-1}O(t_j,z)+2z^2t_k\mathcal{S}(t_1,t_2,\ldots,t_{k-1},z)\\
&\quad+t_k\mathcal{S}(t_1,t_2,\ldots,t_k,z)+2z^2t_k^2\mathcal{S}(t_1,t_2,\ldots,t_k,z),
\end{aligned}$$

which, by (11), is equivalent to

$$\begin{aligned}
\mathcal{S}(t_1,t_2,\ldots,t_k,z) &= \frac{1+2z^2t_k}{1-t_k-2z^2t_k^2}\mathcal{S}(t_1,t_2,\ldots,t_{k-1},z)\\
&\quad+\frac{2^{k-1}z^2\prod_{j=1}^{k-1}(1+z^2t_j)}{\prod_{j=1}^{k}(1-t_j-2z^2t_j^2)} \qquad k\geq 3.
\end{aligned} \tag{12}$$

Using the fact that $\mathcal{S}_{n_1,n_2}(z) = O_{n_1+n_2+2}(z)$, we obtain

$$\begin{aligned}
\mathcal{S}(t_1,t_2,z) &= \sum_{n_1,n_2\geq 0} O_{n_1+n_2+2}(z)t_1^{n_1}t_2^{n_2}\\
&= \sum_{n\geq 2} O_n(z)(t_1^{n-3}+t_1^{n-4}t_2+\cdots+t_1t_2^{n-4}+t_2^{n-3})\\
&= \sum_{n\geq 2} O_n(z)\frac{t_1^{n-2}-t_2^{n-2}}{t_1-t_2}\\
&= \frac{O(t_1,z)-t_1}{t_1^2(t_1-t_2)} - \frac{O(t_2,z)-t_2}{t_2^2(t_1-t_2)}\\
&= \frac{(1+2z^2t_1)(1+2z^2t_2)+z^2(3+2z^2t_1+2z^2t_2)}{(1-t_1-2z^2t_1^2)(1-t_2-2z^2t_2^2)}.
\end{aligned}$$

which, by (11), implies

$$\mathcal{S}(t_1,t_2,z) = \frac{(1+2z^2t_1)(1+2z^2t_2)+z^2(3+2z^2t_1+2z^2t_2)}{(1-t_1-2z^2t_1^2)(1-t_2-2z^2t_2^2)}. \tag{13}$$

Iterating (12) we obtain

$$\begin{aligned}
\mathcal{S}(t_1,t_2,\ldots,t_k,z) &= \mathcal{S}(t_1,t_2,z)\prod_{j=3}^{k}\frac{1+2z^2t_j}{1-t_j-2z^2t_j^2}\\
&\quad + \frac{z^2\sum_{j=3}^k 2^{j-1}\prod_{i=1}^{j-1}(1+z^2t_i)\prod_{i=j+1}^k(1+2z^2t_i)}{\prod_{j=1}^k(1-t_j-2z^2t_j^2)},
\end{aligned}$$

which, by (13), implies the following result.

Theorem 2.5. *Let $k \geq 2$. Then the rank distribution of the overlap matrices for the star-ladder graph $\mathcal{S}_{(n_1,n_2,\ldots,n_k)}$ is given by the generating function*

$$\begin{aligned}
\mathcal{S}(t_1,t_2,\ldots,t_k,z) &= \frac{\prod_{j=1}^k(1+2z^2t_j)}{\prod_{j=1}^k(1-t_j-2z^2t_j^2)}\\
&\quad + \frac{z^2\sum_{j=1}^k 2^{j-1}\prod_{\ell=1}^{j-1}(1+z^2t_\ell)\prod_{\ell=j+1}^k(1+2z^2t_\ell)}{\prod_{j=1}^k(1-t_j-2z^2t_j^2)}.
\end{aligned}$$

Now our aim is to find an explicit formula for $\mathcal{S}_{n_1,n_2,\ldots,n_k}(z)$ by finding the coefficient of $\mathbf{t^n} = t_1^{n_1}t_2^{n_2}\cdots t_k^{n_k}$ in the generating function $\mathcal{S}(t_1,t_2,\ldots,t_k,z)$. At first, note that the coefficient of $\mathbf{t^n}$ in

$$\frac{1}{\prod_{j=1}^k(1-t_j-2z^2t_j^2)}$$

(see Lemma 1.4) is given by

$$
\begin{aligned}
[\mathbf{t}^{\mathbf{n}}]\left(\frac{1}{\prod_{j=1}^{k}(1-t_j-2z^2t_j^2)}\right) &= \prod_{j=1}^{k}[t_j^{n_j}]\left(\frac{1}{1-t_j-2z^2t_j^2}\right)\\
&= \prod_{j=1}^{k}(i\sqrt{2}z)^{n_j}U_{n_j}\left(\frac{1}{2i\sqrt{2}z}\right)\\
&= (i\sqrt{2}z)^{\sum_{j=1}^{k}n_j}\prod_{j=1}^{k}U_{n_j}\left(\frac{1}{2i\sqrt{2}z}\right) \qquad (14)
\end{aligned}
$$

and that

$$\prod_{j=s}^{k}(1+ut_j) = \sum_{A\subseteq[s,k]}u^{|A|}\prod_{a\in A}t_a, \qquad (15)$$

for any $k \geq s$, where $[a,b] = \{a, a+1, \ldots, b\}$.

We define

$$\rho_A(n_j) = (i\sqrt{2}z)^{n_j-\chi_A(j)}U_{n_j-\chi_A(j)}\left(\frac{1}{2i\sqrt{2}z}\right),$$

where $U_n(t)$ is the n^{th} Chebyshev polynomial of the second kind, and $\chi_A(j)$ is defined to be 1 if $j \in A$ or 0 otherwise, and $i^2 = -1$.

Now Theorem 2.5 together with (14) and (15) imply the following result.

Theorem 2.6. *Let $k \geq 2$, let $n_1, n_2, \ldots, n_k \geq 0$. Then the rank-distribution $\mathcal{S}_{n_1,n_2,\ldots,n_k}(z) = [\mathbf{t}^{\mathbf{n}}]\mathcal{S}(t_1, t_2, \ldots, t_k, z)$ is given by the polynomial*

$$
\begin{aligned}
\mathcal{S}_{n_1,n_2,\ldots,n_k}(z) = &\sum_{A\subseteq[1,k]}(2z^2)^{|A|}\prod_{j=1}^{k}\rho_A(n_j)\\
&+ z^2\sum_{j=1}^{k}\sum_{A\subseteq[1,j-1]}\sum_{B\subseteq[j+1,k]}2^{|B|+j-1}z^{2|A|+2|B|}\prod_{j=1}^{k}\rho_{A\cup B}(n_j).
\end{aligned}
$$

Theorem 2.6 reveals the following nice property:

Corollary 2.7. *For $k \geq 2$, let $\pi = (n_1, n_2, \ldots, n_k)$ be a n-tuple of k nonnegative integers, and let π' be any permutation of π. Then $\mathcal{S}_\pi(z) = \mathcal{S}_{\pi'}(z)$.*

Theorem 2.8. *The genus polynomial of the star-ladder SL_U is as follows:*

$$\Gamma_{SL_U}(z) \;=\; 2^{\sum_{j=1}^{k}(n_j+1)}\mathcal{S}_{n_1,n_2,\ldots,n_k}(\sqrt{z}),$$

where $\mathcal{S}_{n_1,n_2,\ldots,n_k}(z)$ is the rank-distribution polynomial defined by equation (8).

Proof. The theorem follows from Proposition 2.3. □

Example 2.9. *Let $k = 3$ and let us find the polynomial $\mathcal{S}_{2,1,0}(z)$. After evaluating each sum in the formula of $\mathcal{S}_{2,1,0}(z)$ according to Theorem 2.6, we obtain*

$$\begin{aligned}\mathcal{S}_{2,1,0}(z) &= (1 + 7z^2 + 12z^4 + 4z^6) + (2z^2 + 6z^4) + (4z^2 + 16z^4 + 12z^6)\\ &= 1 + 13z^2 + 34z^4 + 16z^6.\end{aligned}$$

Thus, $\Gamma_{SL_{2,1,0}}(z) \;=\; 64\mathcal{S}_{2,1,0}(z) = 64 + 832z^2 + 2176z^4 + 1024z^6$.

Example 2.9 can be extended as follows. Let

$$p_n = (i\sqrt{2}z)^n U_n\left(\frac{1}{2i\sqrt{2}z}\right)$$

with $i^2 = -1$. Then Theorem 2.6 for $k = 3$ gives

$$\begin{aligned}\mathcal{S}_{a,b,c}(z) = {} & p_a p_b p_c + 2z^2(p_{a-1}p_b p_c + p_a p_{b-1} p_c + p_a p_b p_{c-1})\\ & + 4z^4(p_{a-1}p_{b-1}p_c + p_{a-1}p_b p_{c-1} + p_a p_{b-1}p_{c-1}) + 8z^6 p_{a-1}p_{b-1}p_{c-1}\\ & + z^2 p_a p_b p_c + 2z^4 p_a p_{b-1} p_c + 2z^4 p_a p_b p_{c-1} + 4z^6 p_a p_{b-1} p_{c-1}\\ & + 2z^2 p_a p_b p_c + 4z^4 p_a p_b p_{c-1} + 2z^4 p_{a-1} p_b p_c + 4z^6 p_{a-1} p_b p_{c-1}\\ & + 4z^2 p_a p_b p_c + 4z^4(p_{a-1}p_b p_c + p_a p_{b-1} p_c) + 4z^6 p_{a-1}p_{b-1}p_c,\end{aligned}$$

which implies this formula

$$\begin{aligned}\mathcal{S}_{a,b,c}(z) = {} & (1 + 7z^2)p_a p_b p_c + 2z^2(1 + 3z^2)(p_{a-1}p_b p_c + p_a p_{b-1} p_c + p_a p_b p_{c-1})\\ & + 4z^4(1 + z^2)(p_{a-1}p_{b-1}p_c + p_{a-1}p_b p_{c-1} + p_a p_{b-1}p_{c-1}) + 8z^6 p_{a-1}p_{b-1}p_{c-1}.\end{aligned}$$

Example 2.10. *Applying this formula for several values of a, b, c we obtain the following values:*

$$\begin{array}{lcl lcl}\mathcal{S}_{0,0,0}(z) & = & 1 + 7z^2 & \mathcal{S}_{1,0,0}(z) & = & 1 + 9z^2 + 6z^4\\ \mathcal{S}_{2,0,0}(z) & = & 1 + 11z^2 + 20z^4 & \mathcal{S}_{1,1,0}(z) & = & 1 + 11z^2 + 16z^4 + 4z^6\\ \mathcal{S}_{3,0,0}(z) & = & 1 + 13z^2 + 38z^4 + 12z^6 & \mathcal{S}_{2,1,0}(z) & = & 1 + 13z^2 + 34z^4 + 16z^6\\ \mathcal{S}_{1,1,1}(z) & = & 1 + 13z^2 + 30z^4 + 20z^6. & & & \end{array}$$

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College of Mathematics and Econometrics, Hunan University, 410082 Changsha, China

E-mail address: `ycchen@hnu.edu.cn`

Department of Computer Science, Columbia University, New York, NY 10027 USA

E-mail address: `gross@cs.columbia.edu`

Department of Mathematics, University of Haifa, 31905 Haifa, Israel

E-mail address: `toufik@math.haifa.ac.il`