Reducing contextual bandits to supervised learning

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Based on joint work with A. Agarwal, S. Kale, J. Langford, L. Li, and R. Schapire

Practicing physician

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Loop:

1. Patient arrives with symptoms, medical history, genome ...

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Goal: prescribe treatments that yield good health outcomes.

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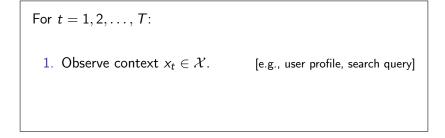
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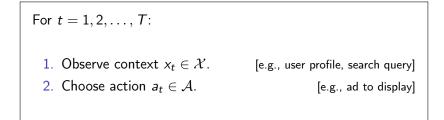
Loop:

- 1. User visits website with profile, browsing history...
- 2. Choose content to display on website.
- 3. Observe user reaction to content (e.g., click, "like").

Goal: choose content that yield desired user behavior.

For
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<u>Contextual</u>: use features x_t to choose good actions a_t . <u>Bandit</u>: $r_t(a)$ for $a \neq a_t$ is not observed.

(Non-bandit setting: whole reward vector $\boldsymbol{r}_t \in [0,1]^{\mathcal{A}}$ is observed.)

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- 3. <u>Selection bias</u>, especially while *exploiting*.

Learning objective

Regret (*i.e.*, relative performance) to a policy class Π :

$$\underbrace{\max_{\pi \in \Pi} \frac{1}{T} \sum_{t=1}^{T} r_t(\pi(x_t))}_{\downarrow = 1} - \underbrace{\frac{1}{T} \sum_{t=1}^{T} r_t(a_t)}_{\downarrow = 1}$$

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Goal: regret $\rightarrow 0$ as fast as possible as $T \rightarrow \infty$.



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New fast and simple algorithm for contextual bandits.

- Operates via reduction to supervised learning (with computationally-efficient reduction).
- Statistically (near) optimal regret bound.

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Suffers from selection bias.

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Never try action B in context X. $\Omega(1)$ regret.

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But perhaps policy class Π has some structure \ldots

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Can often exploit structure of Π to get tractable algorithms.
 Abstraction: arg max oracle (AMO)

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Can't directly use this in bandit setting.

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Regret bound with best τ : ~ $T^{-1/3}$ (sub-optimal).

(Dependencies on $|\mathcal{A}|$ and $|\Pi|$ hidden.)

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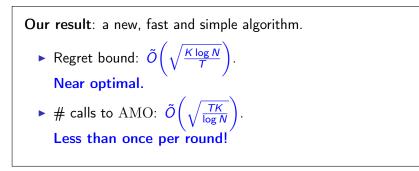
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Monster (Dudik, <u>Hsu</u>, Kale, Karampatziakis, Langford, Reyzin, & Zhang, UAI 2011) Near optimal regret, but $O(T^6)$ calls to AMO.

Our result

Let $K := |\mathcal{A}|$ and $N := |\Pi|$.



Components of the new algorithm:

Importance-weighted LOw-Variance Epoch-Timed Oracleized CONtextual BANDITS

- 1. "Classical" tricks: randomization, inverse propensity weighting.
- 2. Efficient algorithm for balancing exploration/exploitation.
- 3. Additional tricks: warm-start and epoch structure.

1. Classical tricks

What would've happened if I had done X?

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Q: How do I learn about $r_t(a)$ for actions a I don't actually take? A: Randomize. Draw $a_t \sim p_t$ for some pre-specified prob. dist. p_t .

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$$\mathbb{E}_{\boldsymbol{a}_t \sim \boldsymbol{p}_t} \big[\hat{r}_t(\boldsymbol{a}) \big] = \sum_{\boldsymbol{a}' \in \mathcal{A}} p_t(\boldsymbol{a}') \cdot \frac{r_t(\boldsymbol{a}') \cdot \mathbb{1} \{ \boldsymbol{a} = \boldsymbol{a}' \}}{p_t(\boldsymbol{a}')} = r_t(\boldsymbol{a}).$$

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Estimate avg. reward of policy: $\widehat{\text{Rew}}_t(\pi) := \frac{1}{t} \sum_{i=1}^t \hat{r}_i(\pi(x_i)).$

Inverse propensity weighting (Horvitz & Thompson, JASA 1952) Importance-weighted estimate of reward from round *t*:

$$\forall a \in \mathcal{A}. \quad \hat{r}_t(a) := \frac{r_t(a_t) \cdot \mathbb{1}\{a = a_t\}}{p_t(a_t)} = \begin{cases} \frac{r_t(a_t)}{p_t(a_t)} & \text{if } a = a_t \\ 0 & \text{otherwise }. \end{cases}$$

Unbiasedness:

$$\mathbb{E}_{\boldsymbol{a}_t \sim \boldsymbol{p}_t} \big[\hat{r}_t(\boldsymbol{a}) \big] = \sum_{\boldsymbol{a}' \in \mathcal{A}} p_t(\boldsymbol{a}') \cdot \frac{r_t(\boldsymbol{a}') \cdot \mathbb{1}\{\boldsymbol{a} = \boldsymbol{a}'\}}{p_t(\boldsymbol{a}')} = r_t(\boldsymbol{a}).$$

Range and variance: upper-bounded by $\frac{1}{p_t(a)}$.

Estimate avg. reward of policy: $\widehat{\text{Rew}}_t(\pi) := \frac{1}{t} \sum_{i=1}^t \hat{r}_i(\pi(x_i)).$

How should we choose the p_t ?

Hedging over policies

Get action distributions via policy distributions.



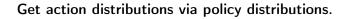
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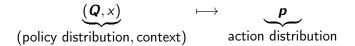
Get action distributions via policy distributions.



Policy distribution: $\mathbf{Q} = (Q(\pi) : \pi \in \Pi)$ probability dist. over policies π in the policy class Π

Hedging over policies





- 1: Pick initial distribution Q_1 over policies Π .
- 2: for round t = 1, 2, ... do
- 3: Nature draws (x_t, \boldsymbol{r}_t) from dist. \mathcal{D} over $\mathcal{X} \times [0, 1]^{\mathcal{A}}$.
- 4: Observe context x_t .
- 5: Compute distribution \boldsymbol{p}_t over \mathcal{A} (using \boldsymbol{Q}_t and x_t).
- 6: Pick action $a_t \sim \boldsymbol{p}_t$.
- 7: Collect reward $r_t(a_t)$.
- 8: Compute new distribution Q_{t+1} over policies Π .

9: end for

2. Efficient construction of good policy distributions

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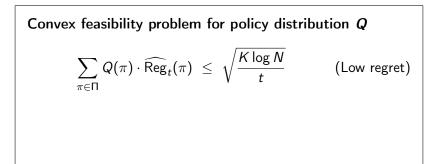
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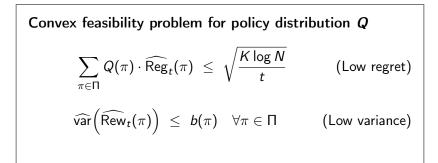
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- 1. Define convex feasibility problem (over distributions Q on Π) such that solutions yield (near) optimal regret bounds.
- 2. Design algorithm that finds a *sparse* solution **Q**.

Algorithm only accesses Π via calls to AMO \implies nnz(Q) = O(# AMO calls)

Convex feasibility problem for policy distribution Q





Convex feasibility problem for policy distribution
$$Q$$

$$\sum_{\pi \in \Pi} Q(\pi) \cdot \widehat{\operatorname{Reg}}_t(\pi) \leq \sqrt{\frac{K \log N}{t}} \qquad \text{(Low regret)}$$

$$\widehat{\operatorname{var}}(\widehat{\operatorname{Rew}}_t(\pi)) \leq b(\pi) \quad \forall \pi \in \Pi \qquad \text{(Low variance)}$$

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Using feasible Q_t in round t gives near-optimal regret. But $|\Pi|$ variables and $>|\Pi|$ constraints, ...

Solver for "good policy distribution" problem

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(If no such violated constraint, stop and return Q.)

(c < 1 and $\alpha > 0$ have closed-form formulae.)

Implementation via AMO

Finding "low variance" constraint violation:

1. Create fictitious rewards for each i = 1, 2, ..., t:

$$\widetilde{r_i}(a) := \widehat{r_i}(a) + rac{\mu}{Q(a|x_i)} \quad \forall a \in \mathcal{A} \,,$$

where $\mu \approx \sqrt{(\log N)/(Kt)}$.

- 2. Obtain $\widetilde{\pi} := \operatorname{AMO}\left(\left\{(x_i, \widetilde{r}_i)\right\}_{i=1}^t\right)$.
- 3. $\operatorname{Rew}_t(\widetilde{\pi}) > \operatorname{threshold}$ iff $\widetilde{\pi}$'s "low variance" constraint is violated.

Iteration bound

Solver is coordinate descent for minimizing potential function $\Phi(\boldsymbol{Q}) := c_1 \cdot \widehat{\mathbb{E}}_x \left[\mathsf{RE}(\mathsf{uniform} \| \boldsymbol{Q}(\cdot | x)) \right] + c_2 \cdot \sum_{\pi \in \Pi} Q(\pi) \widehat{\mathsf{Reg}}_t(\pi).$

(Actually use $(1 - \varepsilon) \cdot \boldsymbol{Q} + \varepsilon \cdot \text{uniform}$ inside RE expression.)

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(Partial derivative w.r.t. $Q(\pi)$ is "low variance" constraint for π .) Returns a feasible solution after

$$\tilde{O}\left(\sqrt{\frac{\kappa t}{\log N}}\right)$$
 steps.

(Actually use $(1 - \varepsilon) \cdot \boldsymbol{Q} + \varepsilon \cdot \text{uniform}$ inside RE expression.)

Algorithm

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Recap



Feasible solution to "good policy distribution problem" gives near optimal regret bound.



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New coordinate descent algorithm:

repeatedly find a violated constraint and adjust ${oldsymbol Q}$ to satisfy it.

Recap

Feasible solution to "good policy distribution problem" gives near optimal regret bound.

New coordinate descent algorithm: repeatedly find a violated constraint and adjust Q to satisfy it.

Analysis: In round t,

nnz(
$$\boldsymbol{Q}_{t+1}$$
) = $O(\# \text{ AMO calls}) = \tilde{O}\left(\sqrt{\frac{Kt}{\log N}}\right)$

.

3. Additional tricks: warm-start and epoch structure

Total complexity over all rounds

In round *t*, coordinate descent for computing \boldsymbol{Q}_{t+1} requires

$$\tilde{O}\left(\sqrt{\frac{\kappa t}{\log N}}\right)$$
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To compute \boldsymbol{Q}_{t+1} in all rounds $t = 1, 2, \ldots, T$, need

$$\tilde{O}\left(\sqrt{\frac{K}{\log N}} T^{1.5}\right) \text{ AMO calls over } T \text{ rounds.}$$

To compute \boldsymbol{Q}_{t+1} using coordinate descent, initialize with \boldsymbol{Q}_t .

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But still need an AMO call to even check if Q_t is feasible!

Regret analysis: Q_t has low instantaneous per-round regret—this also crucially relies on i.i.d. assumption.

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log T updates, so $\tilde{O}(\sqrt{KT/\log N})$ AMO calls overall.

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Squares: only update on rounds $1^2,\,2^2,\,3^2,\,4^2,\,\ldots$

 \sqrt{T} updates, so $\tilde{O}(\sqrt{K/\log N})$ AMO calls per update, on average.

Warm start + epoch trick

Over all T rounds:

- ► Update policy distribution on rounds 1², 2², 3², 4², ..., i.e., total of √T times.
- ► Total # calls to AMO:

$$\tilde{O}\left(\sqrt{\frac{\kappa\tau}{\log N}}\right).$$

AMO calls per update (on average):

$$\tilde{O}\left(\sqrt{\frac{K}{\log N}}\right).$$

4. Closing remarks and open problems



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 Epoch structure allows for policy distribution to change very infrequently; combine with warm start for computational improvements.

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Thanks!

Projections of policy distributions

Given policy distribution Q and context x,

$$orall a \in \mathcal{A}$$
. $Q(a|x) \coloneqq \sum_{\pi \in \Pi} Q(\pi) \cdot \mathbbm{1}\{\pi(x) = a\}$

(so $\boldsymbol{Q} \mapsto \boldsymbol{Q}(\cdot|x)$ is a linear map).

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(so $\boldsymbol{Q} \mapsto \boldsymbol{Q}(\cdot|x)$ is a linear map).

We actually use

$$\boldsymbol{p}_t := \boldsymbol{Q}_t^{\mu_t}(\,\cdot\,|x_t) := (1 - \mathcal{K}\mu_t) \boldsymbol{Q}_t(\,\cdot\,|x_t) + \mu_t \mathbf{1}$$

so every action has probability at least μ_t (to be determined).

The potential function

$$\Phi(\boldsymbol{Q}) := t\mu_t \left(\frac{\widehat{\mathbb{E}}_{x \in H_t} \left[\mathsf{RE}(\mathsf{uniform} \| \boldsymbol{Q}^{\mu_t}(\cdot | x)) \right]}{1 - K\mu_t} + \frac{\sum_{\pi \in \Pi} Q(\pi) \widehat{\mathsf{Reg}}_t(\pi)}{Kt \cdot \mu_t} \right),$$