Exact Inference: Elimination and Sum Product

(and hidden Markov models)

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The first sections of these lecture notes follow the ideas in Chapters 3 and 4 of *An Introduction to Probabilistic Graphical Models* by Michael Jordan. In addition, many of the figures are taken these chapters.

The inference problem

Consider two sets of nodes E and F. We would like to calculate the conditional distribution $p(x_F \mid x_E)$. This is the inference problem.

This amounts to three computations. First we marginalize out the set of variables x_R, which are the variables except x_E and x_F,

$$p(x_E, x_F) = \sum_{x_R} p(x_E, x_F, x_R). \tag{1}$$

From this we then marginalize x_F to obtain the marginal probability of x_E,

$$p(x_E) = \sum_{x_F} p(x_E, x_F). \tag{2}$$

Finally we take the ratio to compute the conditional distribution

$$p(x_F \mid x_E) = p(x_E, x_F)/p(x_E). \tag{3}$$

Our goal is to efficiently compute these quantities.

What is the problem? Suppose R contains many nodes, each taking on one of k values. Then marginalizing them out, in the first calculation, requires summing over $k^{|}R|$ configurations. This will usually be intractable.

Elimination

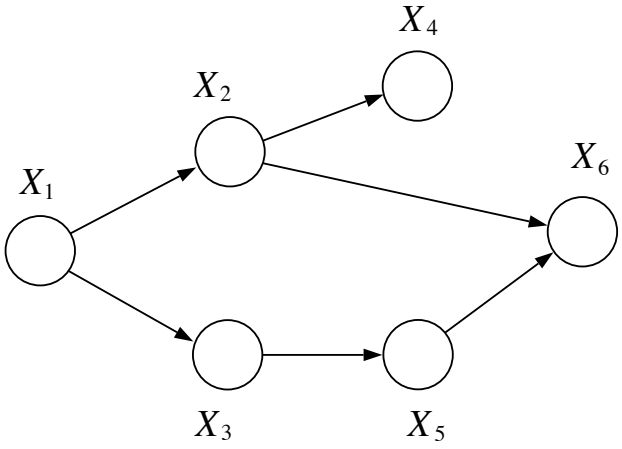
The first algorithm we will discuss is called *elimination*. It is a stepping stone to a more useful algorithm.

First, we introduce some notation. We will need to differentiate between variables that we are summing over or are arguments to a function we are calculating and variables that are clamped at specific values, e.g., because they are part of the evidence.

So, $\bar{x}_6$ will refer to a specific value that the variable x_6 can take on. We will use a delta function, what is called an *evidence potential*, $\delta_{\bar{x}_6}(x_6)$ as a function whose value is equal to one if $x_6 = \bar{x}_6$ and zero otherwise. You will see how this is useful.

An example

We will demonstrate the Eliminate algorithm by an example. Consider the example graphical model from the last lecture,



Let us compute $p(x_1 \mid \bar{x}_6)$.

We multiply the evidence potential at the end of the joint (to clamp x_6) and then compute the marginal $p(x_1, \bar{x}_6)$ by summing out all the variables except x_1.

$$p(x_1, \bar{x}_6) = \sum_{x_2} \sum_{x_3} \sum_{x_4} \sum_{x_5} \sum_{x_6} p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1)p(x_4 \mid x_2)p(x_5 \mid x_3)p(x_6 \mid x_2, x_5)\delta_{\bar{x}_6}(x_6)$$

Normally, one would sum out all except x_1 and x_6, but we use the summation to harness the evidence potential and select the right value of x_6 from the table.[1]

Notice, thanks to the factorization, that we can move some of these summations inside,

$$p(x_1, \bar{x}_6) = p(x_1) \sum_{x_2} p(x_2 \mid x_1) \sum_{x_3} p(x_3 \mid x_1) \sum_{x_4} p(x_4 \mid x_2) \sum_{x_5} p(x_5 \mid x_3) \sum_{x_6} p(x_6 \mid x_2, x_5) \delta_{\bar{x}_6}(x_6)$$

Let's make an intermediate factor involving the summation over x_6,

$$m_6(x_2, x_5) \triangleq \sum_{x_6} p(x_6 \mid x_2, x_5) \delta_{\bar{x}_6}(x_6)$$

Note that this is a function of x_2 and x_5 because they are involved in the terms we summed over. We now rewrite the joint

$$p(x_1, \bar{x}_6) = p(x_1) \sum_{x_2} p(x_2 \mid x_1) \sum_{x_3} p(x_3 \mid x_1) \sum_{x_4} p(x_4 \mid x_2) \sum_{x_5} p(x_5 \mid x_3) m_6(x_2, x_5).$$

We have eliminated x_6 from the RHS calculation.

Let's do the same for the summation over x_5,

$$m_5(x_2, x_3) \triangleq \sum_{x_5} p(x_5 \mid x_3) m_6(x_2, x_5).$$

This is a function of x_2 and x_3. We rewrite the joint,

$$p(x_1, \bar{x}_6) = p(x_1) \sum_{x_2} p(x_2 \mid x_1) \sum_{x_3} p(x_3 \mid x_1) \sum_{x_4} p(x_4 \mid x_2) m_5(x_2, x_3).$$

Notice that the intermediate factor $m_5(x_2, x_3)$ does not depend on x_4. Thus we rewrite the joint again

$$p(x_1, \bar{x}_6) = p(x_1) \sum_{x_2} p(x_2 \mid x_1) \sum_{x_3} p(x_3 \mid x_1) m_5(x_2, x_3) \sum_{x_4} p(x_4 \mid x_2).$$

We continue in this fashion, defining intermediate functions, moving them to the right place in the summation, and repeating. Note that the next function

[1] In more detail, our goal is to have $p(\bar{x}_6 \mid x_5, x_2)$ in the expression. This is achieved with $\sum_{x_6} p(x_6 \mid x_5, x_2) \delta_{\bar{x}_6}(x_6)$. Thus we can treat all of the non-query node variables identically, not differentiating between evidence and non-evidence.

$m_4(x_2)$ actually equals one. But, to keep things programmatic, we will include it anyway.

$$\begin{aligned} p(x_1, \bar{x}_6) &= p(x_1) \sum_{x_2} p(x_2 \mid x_1) \sum_{x_3} p(x_3 \mid x_1) m_5(x_2, x_3) m_4(x_2) \\ &= p(x_1) \sum_{x_2} p(x_2 \mid x_1) m_4(x_2) \sum_{x_3} p(x_3 \mid x_1) m_5(x_2, x_3) \\ &= p(x_1) \sum_{x_2} p(x_2 \mid x_1) m_4(x_2) m_3(x_1, x_2) \\ &= p(x_1) m_2(x_1) \end{aligned}$$

We can further marginalize out x_1 to find the probability $p(\bar{x}_6)$,

$$p(\bar{x}_6) = \sum_{x_1} p(x_1, \bar{x}_6).$$

And finally we can take the ratio to find the conditional probability.

Discussion of the example

Let us ask the question: What drives the computational complexity of the calculation we just made?

In each iteration we form a function by summing over a variable. That function is defined on some number of the other variables. The complexity has to do with forming that function. Functions of few variables are easier to form; functions of many variables are more difficult.

In our example, none of the intermediate functions had more than two arguments. If, for example, each variable is binary then we would never be manipulating a table of more than four items.

[**FOR NEXT TIME**: WE WILL ELIMINATE X_2 FIRST; WHEN DONE CORRECTLY, ELIMINATING X_4 FIRST DOES NOT NECESSARILY LEAD TO WORSE COMPLEXITY.]

Consider this alternative, where we eliminate x_4 first

$$\begin{aligned} p(x_1, \bar{x}_6) &= \sum_{x_2} \sum_{x_3} p(x_1) p(x_2 \mid x_1) p(x_3 \mid x_1) \sum_{x_6} \sum_{x_5} \sum_{x_4} p(x_4 \mid x_2) p(x_5 \mid x_3) p(x_6 \mid x_2, x_5) \delta_{\bar{x}_6}(x_6) \\ &= \sum_{x_2} \sum_{x_3} p(x_1) p(x_2 \mid x_1) p(x_3 \mid x_1) \sum_{x_6} \sum_{x_5} m_4(x_2, x_3, x_5, x_6) \end{aligned}$$

Here the intermediate function is

$$m_4(x_2, x_3, x_5, x_6) \triangleq \sum_{x_4} p(x_4 \mid x_2)\, p(x_5 \mid x_3)\, p(x_6 \mid x_2, x_5)\, \delta_{\bar{x}_6}(x_6)$$

It is still a summation over x_4, but contains many more cells than the first intermediate function from the first example. This is suggestive that the complexity of the algorithm depends on the order in which we eliminate variables.

The elimination algorithm

We will now describe the algorithm. At each iteration, we take the *sum* of a *product* of functions. These functions can be conditional probability tables $p(x_i \mid x_{\pi_i})$, delta functions on evidence $\delta_{\bar{x}_i}(x_i)$, and intermediate functions $m_i(x_{S_i})$. The algorithm maintains an *active list* of functions currently in play.

Our goal is to compute $p(x_F \mid x_E)$.

1. Set an *elimination ordering* I, such that the query node F is last.
2. Set an *active list* of functions. Initialize with
   - Each conditional probability table $p(x_i \mid x_{\pi_i})$
   - Evidence potentials $\delta_{\bar{x}_i}(x_i)$ for each evidence node
3. Eliminate each node i in order:
   (a) Remove functions from the active list that involve x_i.
   (b) Set $\phi_i(T_i)$ equal to the product of these functions, where T_i is the set of all variables involved in them.
   Sum over x_i to compute the intermediate function
   $$m_i(S_i) = \sum_{x_i} \phi_i(T_i).$$
   The arguments to the intermediate function are S_i (i.e., $T_i = S_i \cup i$).
   (c) Put $m_i(S_i)$ on the active list.
4. In the penultimate step, we have the unnormalized joint distribution of the query node and the evidence. Marginalizing out the query node gives us the marginal probability of the evidence.

This algorithm gives us insights into how to use a graph to perform an inference. We defined it for one query node, but it generalizes to computing the conditional distribution of multiple nodes.

Undirected graphs

Recall the semantics of an undirected graph,

$$p(x) = \frac{1}{Z} \prod_{C \in \mathcal{C}} \psi(x_C). \tag{4}$$

The elimination algorithm works on undirected graphs as well. Rather than placing conditional probability tables on the active list, we place the potential functions. See the book for a good example.

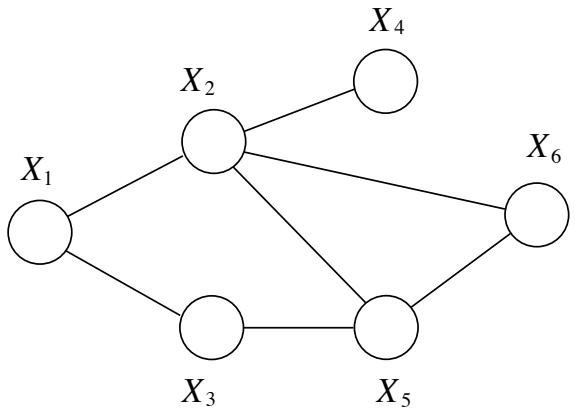
One nuance is the normalizing constant. Reusing the example above, we first compute $p(x_1, \bar{x}_6) = (1/Z)m_2(x_1)$. (This m_2 is different from above, calculated using the undirected elimination algorithm.) We than compute $p(\bar{x}_6) = \sum_{x_1}(1/Z)m_2(x_1)$. Taking the ratio gives the conditional of interest. Note we did not need to compute the normalizing constant Z because it cancels in the ratio. (Question: If we were interested in Z, how would we compute it?)

Graph eliminate

As we saw, the complexity of the elimination algorithm is controlled by the number of arguments in the intermediate factors. This, in turn, has to do with the number of parents of the nodes and the elimination ordering.

We can reason about the complexity of eliminate using only the graph. The idea is that the graph can represent the number of arguments in the intermediate functions and as we eliminate nodes we make sure that it represents the arguments of the intermediate functions.

Let's do the directed case. (The undirected case is easier.) First create an undirected version of the directed graph where we fully connect the parents of each node. (This captures that arguments to $p(x_i \mid x_{\pi_i})$. In our example, here is the so-called "moralized" graph,



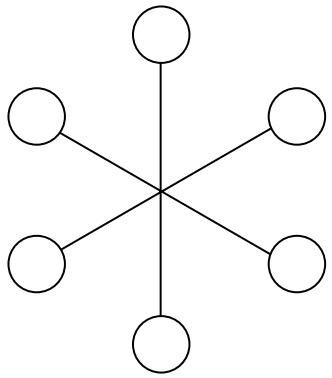
We can run an elimination algorithm graphically. Set the elimination ordering and consider the moralized undirected version of the graphical model. At each iteration, remove the next node and connect the nodes that were connected to it. Repeat.

The *elimination cliques* are the collection of nodes that are the neighbors of x_i at the iteration when it is eliminated. Of course, these are the arguments to the intermediate functions when doing inference. If we record the elimination cliques for an ordering then we see that the complexity of the algorithm is driven by the largest one.

(Formally, the complexity of the elimination algorithm is exponential in the smallest achievable value, over orderings, of the largest elimination clique. However, finding the ordering is an NP-hard problem.)

Examples: Run graph eliminate with $\{6, 5, 4, 3, 2, 1\}$; run graph eliminate with $\{2, 3, 4, 5, 6, 1\}$. Record the elimination cliques.

Example: Consider this hub and spoke graph,



What happens when we remove the center node first; what happens when we remove the leaf nodes first?

Tree propagation

The elimination algorithm gives insight and generalizes to any (discrete) graphical model. But it is limited too, especially because it only computes a single query. We might have multiple inferences to calculate. Further, finding the elimination ordering is a hard problem and has large computational consequences.

We will next discuss the *sum product* algorithm, an inference algorithm for trees.

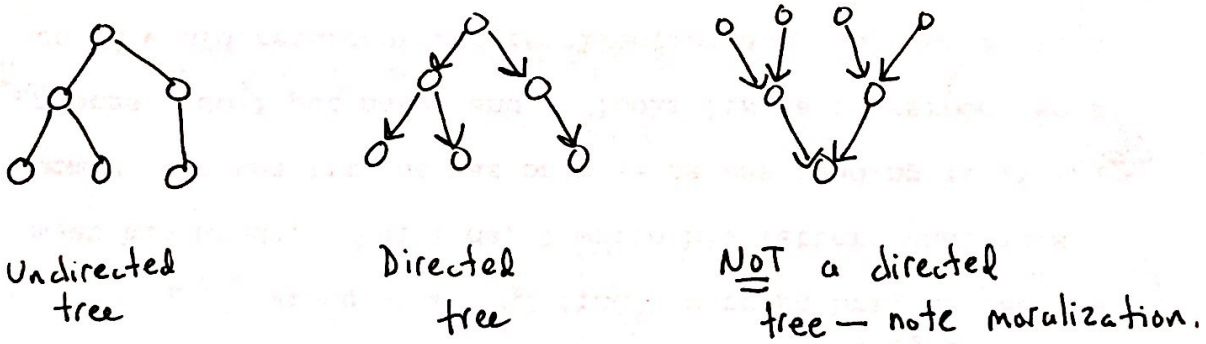
1. Trees are important; many modern graphical models are trees.
2. This is the basis of the *junction tree* algorithm, a completely general exact inference method.
3. It is the basis for approximate inference.

Tree graphical models

An undirected graphical model is a tree if there is only one path between any pair of nodes.

A directed graphical model is a tree if its moralization is an undirected tree, i.e., there are no v-structures in the graph.

Here are some examples



Parameterization. Let's consider undirected trees. We will use a parameterization on singleton "cliques" and pairs (i.e., maximal cliques),

$$p(x) = \frac{1}{Z} \prod_{i \in \mathcal{V}} \psi(x_i) \prod_{(i,j) \in \mathcal{E}} \psi(x_i, x_j). \tag{5}$$

This is parameterized by *singleton* and *pair-wise* potential functions.

Now consider directed trees. We'll see that, for trees, these represent the same class of graphical models. Define the root node to be x_r. The joint in a directed tree is

$$p(x) = p(x_r) \prod_{(i,j) \in \mathcal{E}} p(x_j \mid x_i). \tag{6}$$

Note that we can set potentials equal to these conditional probabilities,

$$\psi(x_r) \triangleq p(x_r) \tag{7}$$

$$\psi(x_i) \triangleq 1 \quad \text{for } i \neq r \tag{8}$$

$$\psi(x_i, x_j) \triangleq p(x_j \mid x_i) \tag{9}$$

The directed tree is a special case of the undirected tree. Any algorithm for undirected trees can be used for the directed tree.

We will only consider undirected trees.

Evidence. Just as we can consider undirected trees without loss of generality, we also will not need to pay special attention to evidence. Consider the evidence nodes x_E. Define

$$\psi^E(x_i) \triangleq \begin{array}{ll} \psi(x_i)\delta_{\bar{x}_i}(x_i) & i \in E \\ \psi(x_i) & i \notin E \end{array}. \tag{10}$$

Notice that

$$p(x \mid \bar{x}_E) = \frac{1}{Z^E} \prod_{i \in \mathcal{V}} \psi^E(x_i) \prod_{(i,j) \in \mathcal{E}} \psi(x_i, x_j) \tag{11}$$

This has the exact same form as the unconditional case.

We will not need to consider if the distribution is conditional or unconditional.

Elimination on trees

First we set an *elimination ordering* of the nodes such that x_1 is the last element. (We will discuss the implications of this ordering later.)

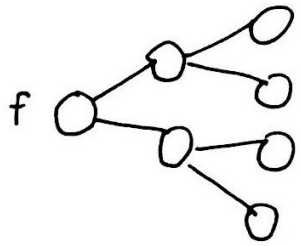
Recall the elimination algorithm:

1. Choose an ordering I such that the query node is last.
2. Place all potentials on the active list.
3. Eliminate each node i
   (a) Take the product of active functions that reference x_i. Pop them off the active list.
   (b) Form the intermediate function $m_i(\cdot)$ by summing over x_i.
   (c) Put the intermediate function on the active list

Now we consider elimination on a tree. Set up the following ordering:

- Treat the query node f as the root.
- Direct all edges to point away from f. (Note this is not a directed GM.)
- Set the ordering such that each node is eliminated only after its children (e.g., order the nodes by depth first).

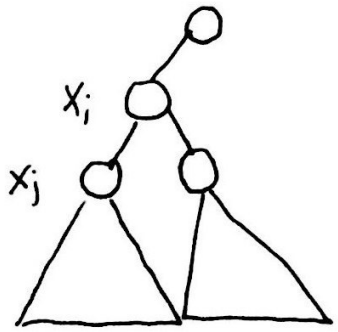
Let's look at a tree and consider the complexity of the elimination algorithm (by running the graph elimination algorithm).



Working backwards from the leaves, we see that the elimination cliques have maximum size 2. Thus this is an efficient algorithm. (Consider removing a child of the root node first to see an inefficient elimination ordering.)

But we will see that elimination on trees leads to a general algorithm for computing the (conditional) probability of any query node.

Let's look at the elimination step in more detail. Consider a pair of nodes (i, j), where i is closer to the root than j.



According to our algorithm, j will be eliminated first.

What is the intermediate function that we create when j is eliminated? To answer this, consider what we know about the potentials that will be on the active list when j is eliminated:

- $\psi(x_j)$
- $\psi(x_i, x_j)$
- no functions including k, a descendant of j
- no functions including ℓ, outside of the subtree of j

Therefore, when x_j is eliminated we add an intermediate function that is *only* a function of x_i. We call this function $m_{j\to i}(x_i)$ as a "message" from j to i.

What will this message be? It will involve

- the singleton potential $\psi(x_j)$
- the pair-wise potential $\psi(x_i, x_j)$
- and the other messages $m_{k\to j}(x_j)$ for other neighbors $k \in \mathcal{N}(j)\backslash i$

The message is

$$m_{j\to i}(x_i) \triangleq \sum_{x_j} \psi(x_j)\psi(x_i, x_j) \prod_{k\in\mathcal{N}(j)\backslash i} m_{k\to j}(x_j). \tag{12}$$

Once we have computed messages up the tree, we compute the final quantity about the query node f,

$$p(x_f \mid \bar{x}_E) \propto \psi(x_f) \prod_{k\in\mathcal{N}(f)} m_{k\to f}(x_f) \tag{13}$$

Note there is no pair-wise potential because f is the root.

Equation 12 and Equation 13 are the elimination algorithm for an undirected tree. As the reading points out, inference involves solving a system of equations where the variables are the messages. The depth-first ordering of the messages ensures that each one is only a function of the messages already computed.

From elimination on trees to the sum-product algorithm

Let's do an example. We have a four node tree and our goal is to compute $p(x_1)$. (Note: there is no evidence.)

We compute the following messages, in this order:

$$m_{3\to 2}(x_2) = \sum_{x_3} \psi(x_3)\psi(x_2, x_3) \tag{14}$$

$$m_{4\to 2}(x_2) = \sum_{x_4} \psi(x_4)\psi(x_2, x_4) \tag{15}$$

$$m_{2\to 1} = \sum_{x_2} \psi(x_2)\psi(x_2, x_1)m_{3\to 2}(x_2)m_{4\to 2}(x_2) \tag{16}$$

$$p(x_1) \propto \psi(x_1)m_{2\to 1}(x_1) \tag{17}$$

Now let's change our query and compute $p(x_2)$. This becomes the root of the tree and we compute messages $m_{1\to 2}$, $m_{3\to 2}$, $m_{4\to 2}$, and finally $p(x_2)$. *Notice that $m_{4\to 2}$ is the same as above.*

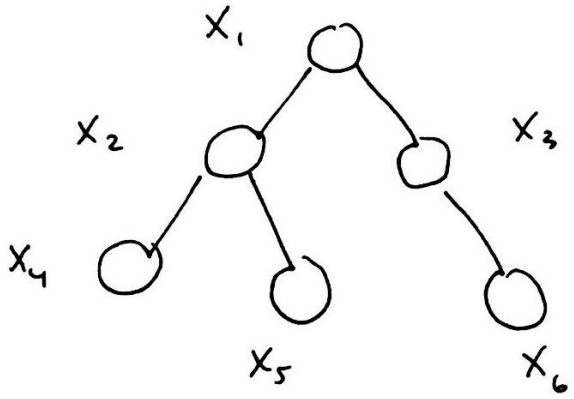
Let's change our query again and compute $p(x_4)$. It becomes the root and we compute messages $m_{1\to 2}$, $m_{3\to 2}$, $m_{2\to 4}$. Again, the messages are reused.

We are reusing computation that we did for other queries. This is the key insight behind the sum-product algorithm.

The sum-product algorithm is based on Equation 12 and Equation 13 and a message passing protocol. To quote from the reading, "A node can send a message to its neighbors when (and only when) it has received messages from all its other neighbors"

This lets us compute any marginal on the graph. It only requires computing 2 messages per node, and each is a function of just one argument.

Consider a six-node tree.



We implement sum-product by propagating messages up and then down the tree. We compute:

- $m_{4\to 2}, m_{5\to 2}, m_{6\to 3}$
- $m_{2\to 1}, m_{3\to 1}$
- $m_{1\to 2}, m_{1\to 3}$
- $m_{2\to 4}, m_{2\to 5}, m_{3\to 6}$

Now we can easily compute any marginal.

The max-product algorithm

Let's consider a different problem, computing the configuration of the random variables that has highest probability. When we condition on evidence, this is called the *maximum a posteriori* problem. It arises in many applied settings.

First, let's look at the simpler problem of finding the maximal achievable probability, $\max p(x)$. Like summation, maximization distributes over addition. Consider (briefly) our old graphical model,

$$\max_x p(x) =$$
$$\max_{x_1} p(x_1) \max_{x_2} p(x_2 \mid x_1) \max_{x_3} p(x_3 \mid x_1) \max_{x_4} p(x_4 \mid x_2) \max_{x_5} p(x_5 \mid x_3) \max_{x_6} p(x_6 \mid x_2, x_5).$$

This looks a lot like computing a marginal. It turns out that all of the derivations that we've made with sums—the elimination algorithm, the sum-product algorithm—can be made with maximums.

Thus, we can define the max elimination algorithm on trees just as we defined the marginal elimination algorithm. (Again, undirected trees subsume directed trees and trees with evidence.) First set a root node. Then define the messages in a depth-first ordering,

$$m^{\max}_{j\to i}(x_i) = \max_{x_j} \psi(x_j)\psi(x_i, x_j) \prod_{k\in\mathcal{N}(j)\setminus i} m^{\max}_{k\to j}(x_j). \tag{18}$$

The last message gives us the maximal probability,

$$\max p(x) = \max_{x_r} \psi(x_r) \prod_{k\in\mathcal{N}(r)} m^{\max}_{k\to r}(x_r) \tag{19}$$

Notice that we do not need to worry about reusing messages here. The maximal probability is the same regardless of the root node.

(Practical note: When computing maximums of probabilities, underflow is a problem. The reason is that when we have many random variables we are multiplying many numbers smaller than one. It's always easier o work in log space. Happily, we can consider $\max \log p(x)$. We define the max-sum algorithm—max takes the place of sum, and sum takes the place of product—and use the log of the potential functions to compute messages.)

We still have not computed the configuration of variables that achieves the maximum. Note that each message tells us what the maximum would be depending on the value of its argument. To get the maximum configuration, we must also store, again for each value of the argument, which value would achieve it. Thus we define another function,

$$\delta_{j\to i}(x_i) \triangleq \arg\max_{x_j} \psi(x_j)\psi(x_i, x_j) \prod_{k\in\mathcal{N}(j)\setminus i} m^{\max}_{k\to j}(x_j). \tag{20}$$

To obtain the maximizing configuration, first compute the messages and δ's going up the tree. Then consider the value of the root that achieves the maximum in Equation 19. Propagate that value as the argument to δ down the tree. This gives the maximizing configuration.

Example: Discrete hidden Markov model

We can now look at a real example, the discrete hidden Markov model. (This is the first real model we will discuss.)

A Markov model, as we've discussed, is a chain of random variables.

[markov model]

This is parameterized by $p(z_{t+1} \mid z_t)$, which is the probability of the next item given the previous item.

A famous example is a *language model*. The random variables are one of some number of terms in a vocabulary. The Markov model specifies the probability of the next term given the previous term. This is called a bigram model. (When there is no connection it is called a unigram model; when there are connections beyond one step back it is called an n-gram model.) Note the parameterization: We must specify a $V \times V$ matrix of probabilities. For each term there is a distribution over the next term.

The *hidden Markov model* is a widely used model that builds on the Markov models assumptions. The idea is that there is a Markov model of hidden random variables—variables that we cannot observe—each of which governs an observation. The hidden variables are called the hidden states.

[hidden markov model from 1 to T, with t-1, t, t+1 in the middle]

In the HMM, the observations are a sequence of items. But what conditional independence assumptions are we making about them? What marginal assumptions are we making? A: They are conditionally independent given the hidden state; they are marginally dependent. You can see this with the Bayes ball algorithm.

What are the parameters to this model? They are the transition probabilities $p(z_{t+1} \mid z_t)$ and the observation probabilities $p(x_t \mid z_t)$. (There is also an initial state probability $p(z_1)$.) Though we have been imagining that each random variable in a graphical model can have its own probability table, typically these values are shared across t. Further, for this discussion we will take the probabilities as fixed and given. We will discuss fitting them in a later lecture.

The inference problem is to compute the marginal of the hidden state or the maximizing sequence of hidden states given a set of observations. Have you seen this before? What are some applications that you know about?

HMMs are everywhere. Here are some examples (from Murphy's book):

- **Speech recognition.** The hidden states are words, governed by a bigram distribution. The observations are features of the audio signal (let's suppose they are discrete), governed by an observation model.

- **Part of speech tagging.** The hidden states are parts of speech, governed by a distribution of the next POS given the previous one. The observations are words themselves. (Question: Suppose there are 10,000 vocabulary words and 5 parts of speech. How does this compare, in terms of the number of probabilities to specify, to the bigram model.)
- **Gene finding.** The observations are a sequence of nucleotides (A,G,C,T) and the hidden states code whether we are in a gene-encoding region of the genome or not. The observation probabilities and transition matrix come from biological knowledge and data.

In all these cases we are interested in computing the marginal probability of a hidden variable (e.g., "What word did she say at time step 4?", "Is the nucleotide at position 600 part of a gene-coding region?") or the maximizing sequence ("What sentence did he most likely just utter?"). These are inference problems, just as we have been discussing.

Notice that the HMM is a tree. We can turn it into an undirected tree with the following potentials:

$$\begin{aligned}
\psi(z_{t-1}, z_t) &\triangleq p(z_t \mid z_{t-1}) \\
\psi(z_t, x_t) &\triangleq p(x_t \mid z_t) \\
\psi(z_1) &\triangleq p(z_1) \\
\psi(z_t) &\triangleq 1, \quad t > 1 \\
\psi(x_t) &\triangleq \delta_{\bar{x}_t}(x_t)
\end{aligned}$$

Now consider the sum-product algorithm. Our goal is to compute any marginal probability of a hidden state. There are a few kinds of messages. Let's start (arbitrarily) at the first time step,

$$\begin{aligned}
m_{x_1 \to z_1}(z_1) &= \sum_{x_1} \psi(x_1)\psi(x_1, z_1) && (21) \\
&= \sum_{z_1} \delta_{\bar{x}_1}(x_1)\, p(x_1 \mid z_1) && (22)
\end{aligned}$$

In general,

$$\begin{aligned}
m_{x_t \to z_t}(z_t) &= \sum_{x_t} \psi(x_t)\psi(x_t, z_t) && (23) \\
&= \sum_{z_t} \delta_{\bar{x}_t}(x_t)\, p(x_t \mid z_t) && (24)
\end{aligned}$$

These messages all select the appropriate observation probability.

Now we can focus on the messages between the hidden states. Beginning with the first time step,

$$m_{z_1 \to z_2}(z_2) = \sum_{z_1} \psi(z_1)\psi(z_1, z_2) m_{x_1 \to z_1}(z_1) \tag{25}$$

$$= \sum_{z_1} p(z_1)\, p(z_2 \mid z_1)\, p(\bar{x}_1 \mid z_1) \tag{26}$$

This lets us move on to the next time step. In general, we can go *forward* from $t-1$ to t with this message,

$$\begin{aligned} m_{z_{t-1} \to z_t}(z_t) &= \sum_{z_{t-1}} \psi(z_t)\psi(z_{t-1}, z_t) m_{x_{t-1} \to z_{t-1}}(z_{t-1}) m_{z_{t-2} \to z_{t-1}}(z_{t-1}) \\ &= \sum_{z_{t-1}} p(z_t \mid z_{t-1})\, p(\bar{x}_{t-1} \mid z_{t-1}) m_{z_{t-2} \to z_{t-1}}(z_{t-1}) \end{aligned}$$

We go *backward* from $t+1$ to t with this message,

$$\begin{aligned} m_{z_{t+1} \to z_t}(z_t) &= \sum_{z_{t+1}} \psi(z_t)\psi(z_t, z_{t+1}) m_{x_{t+1} \to z_{t+1}}(z_{t+1}) m_{z_{t+2} \to z_{t+1}}(z_{t+1}) \\ &= \sum_{z_{t+1}} p(z_{t+1} \mid z_t)\, p(\bar{x}_{t+1} \mid z_{t+1}) m_{z_{t+2} \to z_{t+1}}(z_{t+1}) \end{aligned}$$

This lets us compute any marginal,

$$p(z_t) \propto m_{z_{t-1} \to z_t}(z_t) m_{z_{t+1} \to z_t}(z_t) \tag{27}$$

This is called the forward-backward algorithm or the alpha-beta algorithm. In this case, these messages have probabilistic interpretations. (Exercise: figure them out.) But what is important is that it's an instance of sum-product. While it was derived on its own, we see that it is a special case of a more general algorithm.

Further, especially in speech recognition, we are often interested in the maximizing sequence of hidden states. (E.g., in speech recognition these are the words that maximize the probability of the full utterance.) The max-product algorithm (or max-sum on the log probabilities) gives us this. In the HMM literature, it is known as the Viterbi algorithm. Again, we see that it is a special case of a more general algorithm.